Study of RL-Connectedness and RL-Compactness

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Abstract

In this paper, we introduce a new kind of locally closed sets called regular locally closed sets (briefly RL-closed sets) in a topological space which are weaker than the locally closed sets. Regular locally continuous maps and regular locally irresolute maps are also introduced and studied some of their properties. Finally, we introduce the concept of regular locally connectedness and regular locally compactness on a topological space using the RL-closed sets.

Keywords: RL-closed set, RLC-continuous map, RLC-irresolute map, RL-connectedness, RL-compactness.

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1 Introduction

A topological space is called a connected space if that cannot be represented as the union of two or more disjoint union of non-empty open subsets. Connectedness is one of the principal topological properties that are used to distinguish topological spaces. A subset of a topological space is called a connected set if it is a connected space when viewed as a subspace of that topological space.

In the literature, different type of connectedness were defined and studied by different authors: semipreconnected [1] or β connected [2], preconnected [3], semi-connected [4] using different kind of closed sets. Recently, there were more studies on connectedness and compactness [5,6,7,8]. In this order, we define a new kind of closed sets called RL-closed sets using the concept of locally closed sets and define the connectedness called RL-connectedness and compactness called RL-compactness.

2 Preliminaries

Throughout this paper, we represent X, Y and Z as the topological spaces (X, τ) and (Y, σ) respectively on which no separation axioms are assumed unless otherwise stated. For a subset A of X, $cl(A)$ denotes the closure of A and $int(A)$ denotes the interior of A.

We recall the following definitions in the topological space X.

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Definition 2.1. (9) A subset A of a topological space X is called a regular open set if $A = int(cl(A))$. The complement of the regular open set is called the regular closed set.

Definition 2.2. (10) A subset A of a topological space X is called locally closed set (briefly lc set) if $A = U \cap V$, where U is open and V is closed in (X, τ).

Definition 2.3. A map $f : X \to Y$ is called,

(a) LC-continuous (10) $f^{-1}(V)$ is locally closed set in X for each open set V in Y.

(b) LC-irresolute (10) if $f^{-1}(V)$ is locally closed set in X for each locally closed set V in Y.

3 Regular locally closed sets (RL-closed sets)

Definition 3.1. A subset A of a topological space X is called a regular locally closed set (RL-closed set) if $A = U \cap V$, where U is a regular open set and V is a closed set in (X, τ). The collection of all RL-closed sets of X is denoted by $RLC(X)$.

Example 3.1. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. *Then, the closed sets are* X, $\emptyset$, $\{d\}$, $\{c, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ *and* $RLC(X) = \{X, \emptyset, \{a\}, \{b\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$.

Remark 3.1. It is clear that every RL-closed set is a closed set.

Lemma 3.2. *Every RL-closed set is a locally closed set.*

Proof. Let A be a RL-closed set in a topological space (X, τ). Then, $A = U \cap V$, where U is regular open set and V is closed set in (X, τ). As every regular open set is open, we have A is a locally closed set. □

Proposition 3.1. *If both A and B are RL-closed sets in a topological space (X, τ), then $A \cap B$ is RL-closed set.*

Proof. Let A and B be two RL-closed sets in a topological space X. Then, $A = U_1 \cap V_1$ and $B = U_2 \cap V_2$, where U_1, U_2 are regular open sets and V_1, V_2 are closed sets. Now, $A \cap B = U_3 \cap V_3$, where $U_3 = U_1 \cap U_2$ and $V_3 = V_1 \cap V_2$. Since the intersection of any two regular open sets is regular open set, so we get $A \cap B$ is a RL-closed set. □

Remark 3.2. If both A and B are RL-closed sets in a topological space X, then in general, $A \cup B$ need not be a RL-closed set. This can be seen from the following example:

Example 3.3. *Let* $X = \{a, b, c\}$ *and* $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. *Then*, $RLC(X) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$. *Let* $A = \{a\}$ *and* $B = \{b\}$ *be* RL*-closed sets in* X. *Then,* $A \cup B = \{a, b\}$ *is not a* RL*-closed set in* X.

The regular locally open sets are defined to be the complement of the regular locally closed sets.

Definition 3.2. Let X be a topological space and A be a subset of X. Then, A is called regular locally open set (briefly RL-open set) if $A = U \cup V$, where U is a regular closed set and V is an open set in X. The collection of all regular locally open sets of X is denoted by $RLO(X)$.

Remark 3.3. It is clear that every RL-open set is an open set.

Definition 3.3. Let X be a topological space and let $A \subseteq X$. The union of all RL-open sets contained in A is called regular locally interior of A and is denoted by $int_{RL}(A)$.

Definition 3.4. Let X be a topological space and let $A \subseteq X$. The intersection of all RL-closed supersets of A is called regular locally closure of A and is denoted by $cl_{RL}(A)$. That is, $cl_{RL}(A) = \bigcap_{i \in I} F_i$ where $A \subseteq F_i$, F_i is RL-closed set for each i.

Proposition 3.2. *If both A and B are RL-open sets, then $A \cup B$ is a RL-open set in a topological space X.*

Proof. Let A and B be two RL-open sets in a topological space (X, τ). Then, $A = U_1 \cup V_1$ and $B = U_2 \cup V_2$, where U_1, U_2 are regular closed sets and V_1, V_2 are open sets. Now $A \cup B = U_3 \cup V_3$, where $U_3 = U_1 \cup U_2$ and $V_3 = V_1 \cup V_2$. Since the finite union of regular closed sets is a regular closed set, we get $A \cup B$ is a RL-open set in X. ☐

Definition 3.5. A mapping $f : X \to Y$ is called a regular locally closed continuous map (briefly RLC-continuous map) if $f^{-1}(V)$ is a RL-closed set in X for every closed set V in Y.

Example 3.4. *let $X = Y = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\sigma = \{Y, \emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c$, $f(b) = b$, $f(c) = f(d) = d$. Then, f is a RLC-continuous map.*

Theorem 3.5. *A mapping $f : X \to Y$ is called a RLC-continuous map if and only if the inverse image of every open set in Y is a RL-open set in X.*

Proof. Suppose that $f : X \to Y$ is a RLC-continuous map. Let U be an open set in Y. Then, $Y - U$ is closed in Y. Since f is RLC-continuous, $f^{-1}(Y - U) = X - f^{-1}(U)$ is a RL-closed set in X. Hence $f^{-1}(U)$ is a RL-open set in X.

Conversely, suppose that $f^{-1}(U)$ is a RL-open set in X for every open set U in Y. Let V be a closed set in Y. Then, $Y - V = U$ is an open set in Y so $f^{-1}(U) = X - f^{-1}(V)$ is a RL-open set in X. That is, $f^{-1}(V)$ is a RL-closed set in (X, τ). Hence f is a RLC-continuous map. ☐

Definition 3.6. A mapping $f : X \to Y$ is called regular locally closed irresolute map (briefly RLC-irresolute map) if $f^{-1}(V)$ is a RL-closed set in X for each RL-closed set V in Y.

Example 3.6. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$ and let $Y = \{a, b, c, d\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Define $f : X \to Y$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then, f is a RLC-irresolute map.*

Theorem 3.7. *A mapping $f : X \to Y$ is RLC-irresolute if and only if the inverse image of every RL-open set in Y is a RL-open set in X.*

Proof. Suppose that $f : X \to Y$ is RLC-irresolute map. Let U be a RL-open set in Y. Then, $Y - U$ is RL-closed set in Y so $f^{-1}(Y - U) = X - f^{-1}(U)$ is a RL-closed set in X. Hence $f^{-1}(U)$ is a RL-open set in X.

Conversely, suppose that $f^{-1}(U)$ is a RL-open set in X for every RL-open set U in Y. Let V be a RL-closed set in Y. Then, $(Y - V) = U$ is a RL-open set in Y so $f^{-1}(U) = X - f^{-1}(V)$ is a RL-open set in X. That is, $f^{-1}(V)$ is a RL-closed set in X. Hence f is a RLC-irresolute map. ☐

4 Regular Locally Connected Spaces

Definition 4.1. Let A and B be subsets of a topological space X. Then, A and B are called, RL-separated if $A \cap cl_{RL}(B) = \emptyset = cl_{RL}(A) \cap B$.

Definition 4.2. A topological space X is said to be regular locally connected (briefly RL-connected) if X cannot be written as the union of two non-empty disjoint RL-open sets.

Example 4.1. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Then, the topological space (X, τ) is RL-connected.*

Theorem 4.2. *The following statements are equivalent for a topological space X:*

(a) X is RL-connected,

(b) the only subsets of X which are both RL-open and RL-closed are X and the empty set,

(c) X cannot be expressed as the union of two disjoint non-empty RL-open sets,

(d) there is no RLC-continuous function from X onto a discrete two-point space $\{a, b\}$.

Proof. First we prove (a) $\Rightarrow$ (b). Suppose X is RL-connected and let A be a non-empty RL-open and RL-closed subset of X. Then, $X - A$ is both RL-open and RL-closed. Since X is the disjoint union of RL-open set A and $X - A$, one of these must be empty. That is, $A = \emptyset$ or $X - A = \emptyset$. Now we prove (b) $\Rightarrow$ (c). Suppose that the only subsets of X which are both RL-open and RL-closed are X and the empty set. Let $X = A \cup B$, where A and B are two disjoint non-empty RL-open sets. $B = X - B$, which is a RL-closed set. But this means that B is both RL-open and RL-closed which contradicts our hypothesis. Therefore, X cannot be expressed as the union of two disjoint non-empty RL-open sets. We prove (c) $\Rightarrow$ (d). Suppose that X cannot be expressed as the union of two disjoint non-empty RL-open sets. Let Y be a discrete space with more than one point and let $f : X \to Y$ be an onto RL-continuous function. Define the non-empty RL-open sets U, V such that $Y = U \cup V$. Since f is RLC-continuous, $X = f^{-1}(U) \cup f^{-1}(V)$. But this is a contradiction to the hypothesis. Therefore, Y can't be a discrete space with more than one point. That is, there is no RLC-continuous function from X onto a discrete two-point space $\{a, b\}$. Finally, we prove (d) $\Rightarrow$ (a). If $f : X \to Y$ be an onto RLC-continuous, then $f^{-1}(a)$, $f^{-1}(b)$ are disjoint open subsets of X whose union is X and we have both $f^{-1}(a)$, $f^{-1}(b)$ are non-empty. Then, X is not RL-connected. □

Theorem 4.3. *If $f : X \to Y$ is a RLC-continuous surjective map and X is RL-connected, then Y is connected.*

Proof. Suppose that X is RL-connected and assume that Y is not connected. Then, $Y = A \cup B$, where A and B are non-empty disjoint open sets in Y. Since f is a RLC-continuous surjective map, $X = f^{-1}(A) \cup f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty disjoint RL-open sets. This is a contradiction to that X is RL-connected. Hence Y is connected. □

Theorem 4.4. *If $f : X \to Y$ is a RLC-irresolute surjective map and X is RL-connected, then Y is RL-connected.*

Proof. Suppose that X is RL-connected and assume that Y is not RL-connected. Then, $Y = A \cup B$, where A and B are non-empty disjoint RL-open sets in Y. Since f is a RLC-irresolute surjective map, $X = f^{-1}(A) \cup f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty disjoint RL-open sets. This is a contradiction to that X is RL-connected. Hence Y is RL-connected. □

Definition 4.3. A subset Y of a topological space X is called the RL-subspace of X if $Y \cap U$ is RL-open, when U is RL-open in X.

Definition 4.4. A RL-subspace Y of a topological space X is RL-disconnected if there exist RL-open subsets U and V of X such that $Y \cap U$ and $Y \cap V$ are disjoint non-empty RL-open sets whose union is Y. The RL-subspace is RL-connected if it is not RL-disconnected.

Lemma 4.5. *If Y is a RL-connected subspace of X and if the sets U and V form a RL-separation of X, then $Y \subset U$ or $Y \subset V$.*

Proof. Since U and V are both RL-open in X, the sets $Y \cap U$ and $Y \cap V$ are RL-open in Y. We have, $(Y \cap U) \cup (Y \cap V) = Y$ and $(Y \cap U) \cap (Y \cap V) = \emptyset$. If $Y \cap U$ and $Y \cap V$ are non-empty, then Y is RL-separated. But Y is RL-connected. Then $Y \cap U = \emptyset$ or $Y \cap V = \emptyset$. Therefore, $Y \subset U$ or $Y \subset V$ □

Theorem 4.6. *Let A and B be subspaces of a topological space X. If A and B are RL-connected and not RL-separated, then $A \cup B$ is RL-connected.*

Proof. Assume that $A \cup B$ is not RL-connected. Then, $A \cup B = U \cup V$, where U and V are disjoint non-empty RL-open sets in X. Since A and B are RL-connected, then by Lemma 4.6, either $A \subset U$ or $A \subset V$ and $B \subset U$ or $B \subset V$. If $A \subset U$ and $B \subset U$, then $A \cup B \subset U$ and $V = \emptyset$. This is a contradiction to that V is non-empty. Therefore, $A \cup B$ is RL-connected. □

Theorem 4.7. *If $\{A_\alpha : \alpha \in I\}$ is non-empty collection of RL-connected subspaces of a topological space X such that $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in I} A_\alpha$ is RL-connected.*

Proof. Assume that $Y = \bigcup_{\alpha \in I} A_\alpha$ is not RL-connected. Then $Y = U \cup V$, where U and V are non-empty disjoint RL-open sets in X. Since $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$, there is a point p of $\bigcap_{\alpha \in I} A_\alpha$. Since $p \in Y$, either $p \in U$ or $p \in V$. Suppose that $p \in U$. Since A_α is RL-connected, $A_\alpha \subset U$ or $A_\alpha \subset V$. Since $p \in A_\alpha$, $A_\alpha \not\subset V$. Hence, $A_\alpha \subset U$ for every α. Then $Y = \bigcup_{\alpha \in I} A_\alpha \subset U$. This is a contradiction to that V is non-empty. Therefore, $\bigcup_{\alpha \in I} A_\alpha$ is RL-connected. □

Theorem 4.8. *Let A be a RL-connected subspace of X. If $A \subset B \subset cl_{RL}(A)$, then B is also RL-connected.*

Proof. Assume that B is not RL-connected. Then, $B = U \cup V$, where U and V are disjoint non-empty RL-open sets in B. Since A is RL-connected, then by Lemma 4.6, either $A \subset U$ or $A \subset V$. Suppose that $A \subset U$. Then $cl_{RL}(A) \subset cl_{RL}(U)$. Since $cl_{RL}(U)$ and V are disjoint, B cannot intersect V. This contradicts the fact that V is a non-empty subset of B. Therefore, B is RL-connected. □

Corollary 4.9. *Let $\{A_\alpha : \alpha \in I\}$ be a non-empty collection of RL-connected subspaces of a topological space X and A be a RL-connected subspace of X. If $A \cap A_\alpha \neq \emptyset$ for all α, then $A \cup (\bigcup_{\alpha \in I} A_\alpha)$ is RL-connected.*

Proof. By Theorem 4.7, each set $A \cap A_\alpha$, $\alpha \in I$ is RL-connected and $\bigcap_{\alpha \in I}(A \cap A_\alpha) \neq \emptyset$ since it contains A. Thus, $A \cup (\bigcup_{\alpha \in I} A_\alpha) = \bigcup_{\alpha \in I}(A \cup A_\alpha)$ is RL-connected. □

5 Regular Locally Compact Spaces

Definition 5.1. A collection $\{G_i : i \in I\}$ of RL-open sets of X is said to be RL-open cover for the space X if $X = \bigcup_{i \in I} G_i$.

Definition 5.2. A collection $\{G_i : i \in I\}$ of RL-open sets is said to be RL-open cover for a subset A of the space X if $A \subseteq \bigcup_{i \in I} G_i$.

Definition 5.3. A topological space X is said to be regular locally compact space (briefly RL-compact space) if for every RL-open cover of X has a finite subcover.

Definition 5.4. A subset A of a topological space is said to be regular locally compact set (briefly RL-compact set) if for every RL-open cover of A has a finite subcover.

Theorem 5.1. *Every RL-closed subset of a RL-compact space is a RL-compact space.*

Proof. Let A be a RL-closed set of the RL-compact space X and let $\mathscr{A} = \{G_i : i \in I\}$ be a covering of A by RL-open sets in X. Let $\mathscr{B}$ be a RL-open cover of X. $\mathscr{B} = \mathscr{A} \cup \{X - A\}$. Since X is RL-compact, $\mathscr{B}$ has a finite subcover $\mathscr{B}_{finite}$ of X. If $\mathscr{B}_{finite}$ contains the set $X - A$, discard $X - A$. Otherwise, leave $\mathscr{B}_{finite}$ alone. Then $\mathscr{B}_{finite}$ is a finite sub collection of $\mathscr{A}$ that covers A. □

Theorem 5.2. *The image of a RL-compact space under a RLC-continuous map is compact.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be RLC-continuous map from X onto Y. Let $\{G_i : i \in I\}$ be an open cover for Y. Then, $\{f^{-1}(G_i) : i \in I\}$ is a RL-open cover for X. Since X is RL-compact, this RL-open cover has a finite subcover $\{f^{-1}(G_1), f^{-1}(G_2) \ldots f^{-1}(G_n)\}$. Since f is onto, $\{G_1, G_2 \ldots G_n\}$ is the finite open cover for Y. Therefore, Y is compact. □

Theorem 5.3. *If $f : (X, \tau) \to (Y, \sigma)$ is a RLC-irresolute map and $A \subseteq X$ be a RL-compact relative to X, then the image $f(A)$ is RL-compact relative to Y.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a RLC-irresolute map from topological spaces X onto Y. Let $\{G_i : i \in I\}$ be RL-open cover of $f(A)$ relative to Y. Then, $\{f^{-1}(G_i) : i \in I\}$ is RL-open cover for A relative to X. Since A is RL-compact relative to X, this RL-open cover has a finite subcover $\{f^{-1}(G_1), f^{-1}(G_2) \ldots f^{-1}(G_n)\}$. Since f is onto, $\{G_1, G_2 \ldots G_n\}$ is a finite RL-open cover for $f(A)$. Therefore, $f(A)$ is RL-compact. □

6 Conclusions

A new kind of locally closed sets called RL-closed sets were introduced in topological spaces and some of their main properties were studied. The RL-open sets were defined the complements of the RL-closed sets. Using these RL-open sets, new type of connectedness and compactness called RL-connectedness and RL-compactness were introduced in topological spaces. The RL-connectedness and RL-compactness fulfilled most of the connectedness and compactness properties in topological spaces; therefore, these new constructions can be used to investigate more properties of the topological spaces.

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Competing Interests

Authors have declared that no competing interests exist.

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