

1
2 **Analytical solution of the transport equation for exponentially decreasing**
3 **initial concentration in shallow water table condition in an irrigated field**
4

5
6
7
8

9
10 **Abstract**

11 Most of the agricultural activities are limited for the depth of 15 – 20 cm and rest soil remain enact for long
12 periods which inhibits the microbial activities below this depths and create an initial concentration of
13 nutrients exponentially decreasing with depth. An attempt has been made to develop analytical models for
14 time-dependent nitrification/ denitrification and depth-dependent absorption of urea fertilizer in high water
15 table conditions with fertigation. Laplace transformation method was used to solve the unsteady-state
16 advection-dispersion equation. The analytical solutions that can be derived by this method assist
17 understanding of the movement of fertilizer in irrigated fields. The developed models were validated with
18 the experimental results. They were closely predicting the fertilizer movement in one-dimension soil
19 medium. The little deviation of result from observed values may be due to change of dispersion coefficient
20 and velocity with moisture content. Here these parameters were assumed as constant throughout the time
21 under consideration. Models developed for constant degradation rate is predicting very close to observe
22 values which shows that the soil under study has no depth-dependent degradation. The developed models
23 may be helpful for the planning of drain design, nutrient management and assessment of potential hazards
24 to groundwater in agricultural fields by the knowledge of exact transport parameters and boundary
25 conditions universally.
26

27 **Introduction**

28 The topsoil up to a depth of 15-20 cm supports most of the seasonal and shallow deep-
29 rooted cereals crops. These types of crops may be grown successfully without tillage in a
30 wide range of soil. The popularity of zero-till machines in Indian agriculture avoids the
31 tillage of soil up to a great extent. Addition of different biofertilizers and soil
32 ameliorators increase the biological activities in upper soil and leave the lower soil enact
33 and creates the situation of exponentially decreasing initial concentration of different soil
34 nutrients in soil mass in high water table areas. Mathematical modelling of processes, in
35 the unsaturated zone, is useful for the agricultural management of cultivated sites, for
36 prediction of the fate of agrochemicals, and assessment of the potential hazard of shallow
37 groundwater contamination. The difficulty of solving the transport equation in the
38 unsaturated zone relies on its strong nonlinearity. Although significant efforts have been
39 made to overcome the mathematical difficulties, most analytical solutions are derived for
40 one-dimensional vertical transport under various simplifying assumptions. Accounting
41 for the spatial heterogeneity of natural soils renders the transport problem even more
42 complicated.

43 *Bresler and Laufer* (1974) simulated the movement of nitrate inhomogeneous soil profile
44 in the presence of $\text{NO}_3\text{-N}$ production (nitrification). *Sexton et al.* (1977) modelled nitrate-
45 nitrogen movement and dissipation in fertilized agricultural lands but did not include
46 representation of any other fertilizer from nitrogen. *Wagenet et al.* (1977) extended the
47 mathematical analysis of *Cho* (1971) and *Misra et al.* (1974) to describe the transport and
48 transformation of urea, ammonium nitrogen and nitrate nitrogen soil profile as a function
49 of depth and time subject to either steady or pulse feed application of nitrogen, and
50 validated with controlled laboratory experiments.

51 *Davidson et al.* (1978) developed research and development type models on the fate of
52 nitrogen in the root zone by simplifying assumptions of water and solute processes in the
53 field. *Watts and Hanks* (1978), *Tillotson et al* (1980), *Tillotson and Wagenet* (1982)
54 developed a model that simulated most of the major transformations of the nitrate as well
55 as the uptake by the crop, but fell short of fully describing the system in the plant growth
56 and yield response. *Selim and Iskander* (1981) developed the model for calculating
57 pollution from organic wastes and excessive fertilization. *Tanji et al.* (1982) presented a
58 steady-state nitrogen model developed on a mass balance approach, which considered
59 water and nitrogen flow on an annual or cropped cycle time basis. *Barraclough* (1989),
60 *Borg et al.* (1990), *Benbi et al.* (1991), *Carbon et al.* (1991) also developed soil water,
61 nitrogen models. *Izadi et al.* (1996) combined functional sub-model and analytical
62 solution to the steady-state convection dispersion equations to predict the nitrate
63 concentration in the root zone. *Lessoff and Indelman* (2004) investigated the penetration
64 of reactive solute into a soil during a cycle of water infiltration and redistribution by
65 deriving analytical closed-form solutions for fluid flux, moisture content and contaminant
66 concentration *Sander and Braddock* (2005) presented a range of analytical solutions to
67 the combined transient water and solute transport for horizontal flow. *Smedt* (2007)
68 reported an analytical solution and analysis of solute transport in rivers affected by
69 diffusive transfer in the hyporheic zone. *Khakpour and Kambiz* (2008) reported an
70 analytical solution of transport phenomena within the PEM fuel cell. *Zhan et al* (2009)
71 deduced an analytical solution of two-dimensional solute transport in an aquifer-aquitard
72 system. *Srinivasan and Clement* (2008) reported analytical solutions for sequentially
73 coupled one-dimensional reactive transport problems. *Sadek* (2009) compared the
74 various available analytical solution with numerical methods is deduced that the
75 analytical solution may be used as a versatile tool for assessment of contaminant
76 transport. *Jozse and Janos* (2009) derived an analytical solution of the coupled 2-D
77 turbulent heat and vapour transport equations and the complementary relationship of
78 evaporation. *Guerrero and Skaggs* (2010) presented a general analytical solution for
79 linear, one-dimensional advection-dispersion equation with distance-dependent
80 coefficients. An integrating factor was employed to obtain a transport equation that has a
81 self-adjoint differential operator, and a solution was found using the generalized integral
82 transform technique.

83

84 The mechanisms of solute transport in the irrigated field are significantly influenced
85 by attenuation processes such as adsorption and nitrification/denitrification processes.
86 Most of the available analytical solutions are based on linear equilibrium adsorption and
87 first-order nitrification and possibly zeroth-order production (*van Genuchten and Alves*
88 (1982) for several analytical solutions). Here the movement of urea fertilizer was
89 analytically solved under depth-dependent adsorption factor and combination of constant
90 and exponential nitrification/denitrification rate for exponentially decreasing initial
91 condition. Following assumptions were considered for formulating the boundary value
92 problems:

- 93 1. The soil is unconfined, homogeneous and isotropic overlying an impermeable layer
94 which is having water table depth H meter from the soil surface,
- 95 2. The water through deep percolation moves vertically downward until it joins the
96 groundwater,

97 3. Darcy and Fick's laws hold good,
 98 4. The fluid is of constant density and viscosity,
 99 In the present study 1-D, Richard's equation in combination with solute transport
 100 equation, which incorporates nitrification and de-nitrification, and depth-dependent soil
 101 and water matrix factor was used to characterize the movement of applied fertilizer in
 102 irrigated agriculture having shallow water table conditions.

103 **Governing Equation**

104 Transport equation in the unsaturated porous medium is given by:

105

$$106 \quad \frac{\partial}{\partial t}(\theta C + \rho S) = \frac{\partial}{\partial z} \left(\theta D \frac{\partial C}{\partial z} - qC \right) - \alpha \theta C - \rho \beta C \mp \gamma \theta \quad \dots(1)$$

107 where $C = C(z, t)$ is the concentration of chemical in the liquid phase in mg/l, $S =$
 108 $S(z, t)$ is the concentration of chemical in the solid phase in mg/l, $D = D(z, t)$ is the
 109 dispersion coefficient in m^2/day , $\theta = \theta(z, t)$ is the volumetric water content cm^3
 110 $/\text{cm}^3$, $q = q(z, t)$ is the flux of water in m/day, $\rho = \rho(z)$ is the soil bulk density
 111 in gm/cm^3 , $\alpha = \alpha(z)$ is the first-order degradation rate constant in the liquid
 112 phase, $\beta = \beta(z)$ is the first-order degradation rate constant in the solid phase,
 113 $\gamma = \gamma(z)$ is the zero-order production rate constant in the liquid phase.

114 Here α, β, γ and are zero or greater.

115 Considering that soil medium remains intact with time, and introducing mass balance
 116 equation for one-dimensional unsteady unsaturated flow condition as given by Chow *et*
 117 *al.* (1988), Eqn. (1) reduces to:

118

$$119 \quad \frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial z^2} - v \frac{\partial C}{\partial z} - RC \mp \gamma \quad \dots(2)$$

120 where $v = q/\theta$ $R = \left(\alpha + \frac{\rho\beta}{\theta} \right)$ and, the factor representing the combined effect of liquid

121 and solid phase degradation rate. Here we assume $R(C) = R_0 - b(C_0 - C)$ where R_0
 122 represents potential degradation rate at the land surface; b is reduction factor due to
 123 which degradation decreases linearly as the depth from the land surface it increases up to
 124 a specific value; and C_0 is initial concentration at the ground surface. For the
 125 development of the model, the combination of constant and exponentially decreasing
 126 nitrification/de-nitrification rate which may be given as $\gamma(t) = \gamma_0 + \gamma_1 e^{-rt}$, where γ_0, γ_1 and
 127 are constant nitrification/de-nitrification rates, r is decay constant and t represents time.

128

129 The initial and boundary conditions in mathematical terms, for the solute flow problem in
 130 the unsaturated zone under the above situation, may be written as:

$$\left. \begin{aligned}
& C(z,0) = C_z(z,0) \quad \text{at } t = 0 \quad \text{for } 0 < z < H \\
131 \quad & C(0,t) = C_1 \quad \text{at } t > 0 \quad \text{for } z = 0 \\
& C(H,t) = C_2 \quad \text{at } t > 0 \quad \text{for } z = H
\end{aligned} \right\} \dots(3)$$

132 where $C_1 = g_1 e^{-ht}$, $C_2 = g e^{ht}$, g_1 and g are the concentrations at the ground surface and H
133 meter below the soil surface before application of fertigation. $C_z(z,0)$ is the distribution
134 of initial concentration in the porous medium. Devising a transform given by Eqn. (4)
135 converted the Eqn. (2) and Eqn. (3) into standard heat flow equation and given by Eqn (5)
136 and Eqn(6), respectively.

$$137 \quad C(z,t) = V(z,t) \exp\left(\frac{vz}{2D} - \left(\frac{v^2}{4D} + b\right)t\right) + \frac{\gamma_1 e^{-rt}}{b-r} + \frac{\gamma_0 + bC_0 - R_0}{b} \dots(4)$$

138

$$139 \quad \frac{\partial^2 V}{\partial z^2} = \frac{1}{D} \frac{\partial V}{\partial t} \dots(5)$$

$$\left. \begin{aligned}
& V(z,0) = (C_z - A - B)e^{-az} = f(z) \\
140 \quad & V(0,t) = (C_1 - A - B e^{-rt})e^{dt} = f_1(t) \\
& V(H,t) = E(C_2 - A - B e^{-rt})e^{dt} = f_2(t)
\end{aligned} \right\} \dots(6)$$

$$141 \quad \text{where, } A = \frac{\gamma_0 + bC_0 - R_0}{b} \quad d = \frac{v^2}{4D} + b \quad a = \frac{v}{2D} \quad E = e^{-aH} \quad \text{and} \quad B = \frac{\gamma_1}{b-r}$$

142

143 The general solution of transformed Eqn (5) under initial and boundary condition Eqn(6)

144 is given by *Carslaw* and *Jaeger* (1959) and *Ozisik* (1980) as below:

$$\begin{aligned}
145 \quad V(z,t) &= \frac{2}{H} \sum_{m=1}^{\infty} e^{-D\beta_m^2 t} \sin \beta_m z \left[\int_0^H f(z) \sin \beta_m z dz \right] + \left(1 - \frac{z}{H}\right) f_1(t) + \frac{z}{H} f_2(t) \\
146 \quad &- \frac{2}{H} \sum_{m=1}^{\infty} \frac{\sin \beta_m z}{\beta_m} \left[f_1(0) e^{-D\beta_m^2 t} + \int_0^t e^{-D\beta_m^2(t+\tau)} df_1(\tau) \right] \\
147 \quad &+ \frac{2}{H} \sum_{m=1}^{\infty} (-1)^m \frac{\sin \beta_m z}{\beta_m} \left[f_2(0) e^{-D\beta_m^2 t} + \int_0^t e^{-D\beta_m^2(t+\tau)} df_2(\tau) \right] \dots(7)
\end{aligned}$$

148 where β_m is the root of $\sin \beta_m H = 0$ and τ is a dummy variable. The solution of the
 149 transport equation was obtained for exponentially decreasing initial concentration of
 150 nitrogen in soil profile with the help of equation (7) and transformed initial and boundary
 151 conditions. When $C_z = p - e^{kz}$ i.e. exponentially decreasing with depth, then the final
 152 solution of Eqn. (7) takes the following form:

$$\begin{aligned}
 153 \quad C(z,t) &= \frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} A_2 \sin \beta_m z \left[(p - A - B) \frac{\beta_m}{a^2 + \beta_m^2} \left[1 - (-1)^m E \right] \right] \\
 154 \quad &- \frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} A_2 \sin \beta_m z \frac{\beta_m}{(k-a)^2 + \beta_m^2} \left[1 - (-1)^m E e^{kH} \right] \\
 155 \quad &+ \left(1 - \frac{z}{H} \right) (C_1 - A - B e^{-rt}) e^{az} + \frac{z}{H} (C_2 - A - B e^{-rt}) e^{a(z-H)} \\
 156 \quad &- \frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} \frac{\sin \beta_m z}{\beta_m} [(g_1 - A - B) A_2] \\
 157 \quad &- \frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} A_2 \frac{\sin \beta_m z}{\beta_m} \left[\frac{A_6 A_2}{s} (e^{-st} - 1) - \frac{A_4 A_2}{K} (e^{-Kt} - 1) - \frac{A_2 B_1}{N} (e^{-Nt} - 1) \right] \\
 158 \quad &+ \frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} (-1)^m A_2 \frac{\sin \beta_m z}{\beta_m} [(g - A - B) E] + A + B e^{-rt} \\
 159 \quad &+ \frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} (-1)^m A_2 E \frac{\sin \beta_m z}{\beta_m} \left[\frac{A_3}{l} (e^{-lt} - 1) - \frac{B_1}{N} (e^{-Nt} - 1) - \frac{A_4}{K} (e^{-Kt} - 1) \right] \dots(8)
 \end{aligned}$$

$$160 \quad A_2 = e^{-D\beta_m^2 t} \quad A_3 = (h+d)g \quad A_4 = d \quad A_5 = (C_0 - A - B)$$

$$161 \quad A_5 = (C_0 - A - B) \quad A_6 = g_1(d-h), \quad B_1 = B(d-r)$$

$$162 \quad K = D\beta_m^2 t + d, \quad s = D\beta_m^2 t + d - h \quad l = D\beta_m^2 t + d + h \quad N = D\beta_m^2 t + d - r,$$

163 When degradation is constant with depth i.e. $b=0$ Eqn (8) become imperative so for this
 164 situation another transformation equation (Eqn.9) was devised to transform the original
 165 problem into standard heat flow equation and given as

$$166 \quad C(z,t) = V(z,t) \exp \left(\frac{vz}{2D} - \left(\frac{v^2}{4D} \right) t \right) + \frac{\gamma_1 e^{-rt}}{r} + (\gamma_0 + R_0) t \dots(9)$$

167 This transformation Eqn. (9) transform the problem into a simple heat flow equation
 168 under a constant degradation rate and gave the final solution of the problem as:

$$\begin{aligned}
169 \quad C(z,t) &= \frac{2}{H} e^{az-d_1t} \sum_{m=1}^{\infty} A_2 \sin \beta_m z (p - B_2) \frac{\beta_m}{a^2 + \beta_m^2} [1 - (-1)^m E] \\
170 \quad &- \frac{2}{H} e^{az-d_1t} \sum_{m=1}^{\infty} A_2 \sin \beta_m z \frac{\beta_m}{(k-a)^2 + \beta_m^2} [1 - (-1)^m E e^{kH}] \\
171 \quad &+ \left(1 - \frac{z}{H}\right) (C_1 - A_1 - B_2 e^{-rt}) e^{az} + E \cdot \frac{z}{H} (C_2 - A_1 - B_2 e^{-rt}) e^{az} \\
172 \quad &- \frac{2}{H} e^{az-d_1t} \sum_{m=1}^{\infty} \frac{\sin \beta_m z}{\beta_m} (g_1 - B_2) A_2 + B_2 + A_1 t \\
173 \quad &- \frac{2}{H} e^{az-d_1t} \sum_{m=1}^{\infty} \frac{\sin \beta_m z}{\beta_m} A_2 \left[\begin{array}{c} \frac{A_7 (e^{S_1 t} - 1)}{S_1} - \frac{B_3 (e^{N_1 t} - 1)}{N_1} - \frac{A_1 (e^{K_1 t} - 1)}{K_1} \\ - \frac{A_8}{K_1^2} + \frac{A_8 e^{K_1 t}}{K_1^2} - \frac{A_8 t e^{K_1 t}}{K_1} \end{array} \right] \\
174 \quad &+ \frac{2}{H} e^{az-d_1t} \sum_{m=1}^{\infty} (-1)^m \frac{\sin \beta_m z}{\beta_m} (g - B_2) E A_2 \\
175 \quad &+ \frac{2}{H} e^{az-d_1t} \sum_{m=1}^{\infty} (-1)^m \frac{\sin \beta_m z}{\beta_m} A_2 E \left[\begin{array}{c} \frac{A_9 (e^{-M t} - 1)}{M} - \frac{B_3 (e^{-N_1} - 1)}{N_1} - \frac{A_1 (e^{K_1 t} - 1)}{K_1} \\ - \frac{A_8}{K_1^2} + \frac{A_8 e^{K_1 t}}{K_1^2} - \frac{A_8 t e^{K_1 t}}{K_1} \end{array} \right] \quad \dots(10)
\end{aligned}$$

$$176 \quad A_1 = (\gamma_0 + R_0), \quad A_7 = g_1(d_1 - k) \quad B_3 = B_2(d_1 - r) \quad A_8 = A_1 d_1$$

$$177 \quad B_2 = \frac{\gamma_1}{r} \quad d_1 = \frac{v^2}{4D} \quad S_1 = D\beta_m^2 + d_1 - k, \quad K_1 = D\beta_m^2 + d_1$$

$$178 \quad N_1 = D\beta_m^2 + d_1 - r \quad M = D\beta_m^2 + d_1 + k$$

179 Equation (8) and equation (10) give the complete solution of transport equation (2) under
180 constant and depth-dependent degradation rate for the combination of constant and
181 exponentially denitrification rate. In further analysis, they would be treated as Model 1
182 and Model 2, respectively.

183 **Experimental plot:** The size of the experimental plot was 5 m x 5 m, surrounded by 1-
184 meter buffer zone earlier used by *Behera* (2003), and *Garg et al.* (2005) and lined by a
185 galvanized iron sheet as discussed by *Jaynes et al.* (1992). The line of tensiometers and
186 soil-water samplers were put 1.5 away from the side boundary, double ring infiltrometer
187 was kept at the centre of the plot while access tubes were installed on the centre line of
188 the plot. Depth of both tensiometers and samplers were kept 15, 30, 50, 75, 100 and 150
189 cm below the ground surface. First and sixth were installed 50 cm away from the

190 boundary and the distance between two were kept 80 cm. access tubes were installed 125
 191 cm from the boundary. Observation wells were installed at two corners diagonally,
 192 keeping in mind the general flow direction of water movement. All soil water samplers
 193 were connected by a lateral line through HDPP (high-density polyvinyl pipe) and
 194 connected to a vacuum pump which creates suction and pressure in a sampler for
 195 collection of leachate sample.

196

197 **Collection of field data:** Nitrogen solution of 448 ppm concentration, representing the
 198 nitrogen dose of 334 N kg/ ha, was applied in the experimental plot instantaneously to
 199 simulate the fertigation. Leachate samples were collected with the help of soil-water
 200 sampler and vacuum pump. Collected samples were brought to the laboratory and
 201 analyzed for total nitrogen content with the help of *Kjeldahl* unit.

202 **Result and Discussion**

203 **Verification of the analytical solution with experimental**

204 Physical, chemical, textural and transport parameters, required to validate the developed
 205 models, were obtained by standard methods. Computer programmes for model-1 and
 206 model-2 were developed in C++ language with defining the all input parameters in the
 207 programme except space and time. Just by giving the value of space and time one can get
 208 the concentration of fertilizer at that space and time. Performance of developed models
 209 was compared with experimental results and shown in Fig.-1 to Fig.13. First 6 figures are
 210 showing the performance of developed models at 0.15 m, 0.30 m, 0.50 m, 0.75 m, 1.00 m
 211 and 1.50 m respectively. At 0.15 m first, four days both model-1 and model-2 were over
 212 predicting a little more than observed value but from the third day onwards both
 213 predicted very close to the observed values which may be seen in Fig.1. Similar
 214 performance of models was also observed for other depths except for 1.5 m and is
 215 depicted in Fig.-2 to Fig.-5 that may be due to preferential flow (funneling, fingering
 216 and channelling) of water through the soil or highly disturbed upper soil layer during the
 217 installation of soil-water sampler or combination of these two. Similar performance of
 218 models was also depicted in Fig.-7 to Fig.-13 at a different day and further validated
 219 their performance. Table 1 shows the per cent deviation of concentration predicted by
 220 developed models and observed values. The deviation is very less except for the first two
 221 days.

222 Table-1: Observed and predicted concentration (ppm) by equation 8 and equation 10

Time(days)	Model1	Model 2	Observed	% deviation	% deviation
1	409.46	395.2	375	9.19	5.39
2	470.26	464.23	451	4.27	2.93
3	481.4	477.44	473	1.78	0.94
4	483.73	480.1	478	1.20	0.44
5	484.34	480.76	481	0.69	-0.05
7	484.84	480.83	482	0.59	-0.24
10	485.64	480.73	483	0.55	-0.47

223

224 **Limiting conditions**

225 Analytical solutions are given by Eqns. (8) and (10) under different conditions can be
226 used to obtain the following analytical solutions as special cases: (1) Analytical solutions
227 when the nitrification rate is constant by substituting $\gamma_1 = 0$ in the above equations.
228 Graphical comparison of developed models with observed value for this condition is
229 shown in Fig- 14 to Fig- 26. (2) Analytical solutions when the nitrification rate is
230 exponentially decreasing by substituting $\gamma_0 = 0$ in the above equations. Graphical
231 comparison of developed models with observed value for this condition is shown in Fig
232 27 to Fig- 39. (3) Analytical solutions when there is no nitrification by substituting
233 γ_0 and $\gamma_1 = 0$ in the above equations. Graphical comparison of developed models with
234 observed value for this condition is shown in Fig- 39 to Fig- 52 and (4) Analytical
235 solutions for non-absorbing solutes by substituting R_0 and $b = 0$ in the above equations.
236 Variations in concentrations under limiting conditions were negligible for model 2 as
237 compared to model 1 in similar situations. Model-2 performed better than Model-1 at
238 each day. Hence it may be concluded that the under local soil condition there is no
239 degradation with depth for nitrogen concentration in shallow groundwater table
240 condition.

241

242 **Conclusion**

243 Developed models would be successfully used for the prediction of fertilizer movement
244 in the irrigated field where the water table is high with the accurate knowledge of local
245 transport parameters. Deviation in observed and predicted concentrations were highest on
246 the first day and decreases continuously as time passes this may be due to the highly
247 disturbed top layer caused due to installation of instruments and G.I. sheet. Hence, the
248 preferential flow of solute must be minimized before taking the actual observation to
249 avoid such an outcome.

250

251 **References**

252 Behera, S., Jha, M. K. and Kar, S. (2003). Dynamics of water flow and fertilizer solute
253 leaching in lateritic soils of Kharagpur region, India. *Agricultural Water Management*. 63
254 77–98.

255

256 Benbi D K; Prihar S S; Cheema H S (1991). A model to predict changes in soil moisture,
257 NO_3^- -N content and N uptake by wheat. *Fertilizer Research*, 28(1), 73-84

258

259 Bresler E; Laufer A (1974). Simulation of nitrate movement in soil under transient
260 unsaturated flow condition. *Israel Journal Agriculture Research*, 23, 141-153

261

262

263 Carslaw, H. S. and Jaeger, J.C. (1959). *Conduction of heat in solids*. Clarendon press,
264 London.

265

266 Chow, V.T. Maidment, D.R. and Mays L.W. (1988). *Applied Hydrology*. McGraw-Hill
267 International, New York.

268
269 Domenico, P., 1987. An analytical model for multidimensional transport of a decaying
270 contaminant species. *J. Hydrology*. 91:49-58.
271
272 Garg, K. K.; Jha, M. K. and Kar, S.(2005). Field Investigation of Water Movement and
273 Nitrate Transport under Perched Water Table Conditions. *Bio-Systems Engineering*, 92
274 (1): 69–84.
275
276 Guerrero J.S. Pérez, T.H. Skaggs(2010)Analytical solution for one-dimensional
277 advection–dispersion transport equation with distance-dependent coefficients *Journal of*
278 *Hydrology, Volume 390, Issues 1-2, 20 August 2010, Pages 57-65*
279
280 Hongbin Zhan, Zhang Wen, Guanhua Huang, Dongmin Sun (2009) Analytical solution of
281 two-dimensional solute transport in an aquifer–aquitard system *Journal of Contaminant*
282 *Hydrology, Volume 53, Issues 1-2, 1 December 2001, Pages 41-61.*
283
284 Izadi B; Ashraf M S; Studer D; McCann I; King B (1996). A simple model for the
285 prediction of nitrate concentration in the potato root zone. *Agricultural Water*
286 *Management*, 30, 41 - 56
287
288 Jaynes, D.B., Rice, R.C. and Hunsaker D.J.(1992). Solute transport during chemigation
289 of a level basin. *ASAE Transaction*, 36:1809-1815.
290
291 Jozsef Szilagyi and Janos Jozsa (2009)Analytical solution of the coupled 2-D turbulent
292 heat and vapor transport equations and the complementary relationship of
293 evaporation *Journal of Hydrology, Volume 372, Issues 1-4, 15 June 2009, Pages 61-67*
294
295 Levenspiel, O. (1972). *Chemical reaction Engineering* (2nd ed.) New York: Wiley.
296
297 Mehrzad Khakpour and Kambiz Vafai (2008). Analysis of transport phenomena within
298 PEM fuel cells – An analytical solution. *International Journal of Heat and Mass*
299 *Transfer, Volume 51, Issues 15-16, 15 July 2008, Pages 3712-3723.*
300
301 Misra C; Nielsen D R; Bigger JW(1974). Nitrogen transformation in soil during leaching:
302 ii. steady state nitrification and nitrate reduction. *Soil Science Society American*
303 *Proceeding*, 38, 294-299
304
305 Ozisik, M.N.(1980). *Heat Conduction*. John Wiley and Sons, New York.
306
307 Sander G.C. and R.D. Braddock . Analytical solutions to the transient, unsaturated
308 transport of water and contaminants through horizontal porous media *Advances in*
309 *Water Resources* (2005)
310
311 Sadek (2009) Comparison between numerical and analytical solution of solute transport
312 models. *Journal of African Earth Sciences, Volume 55, Issues 1-2, September 2009,*
313 *Pages 63-68*

314
315
316
317
318
319
320
321
322
323
324
325
326
327
328
329
330
331
332
333
334
335
336
337
338
339
340
341
342
343
344

Saxton K E; Schuman G E; Burwell R E (1977). Modelling nitrate movement and dissipation in fertilized soils. *Soil Science Society American Journal*, 41, 265-273.

Smedt (2007) Analytical solution and analysis of solute transport in rivers affected by diffusive transfer in the hyporheic zone *Journal of Hydrology, Volume 339, Issues 1-2, 10 June 2007, Pages29-38.*

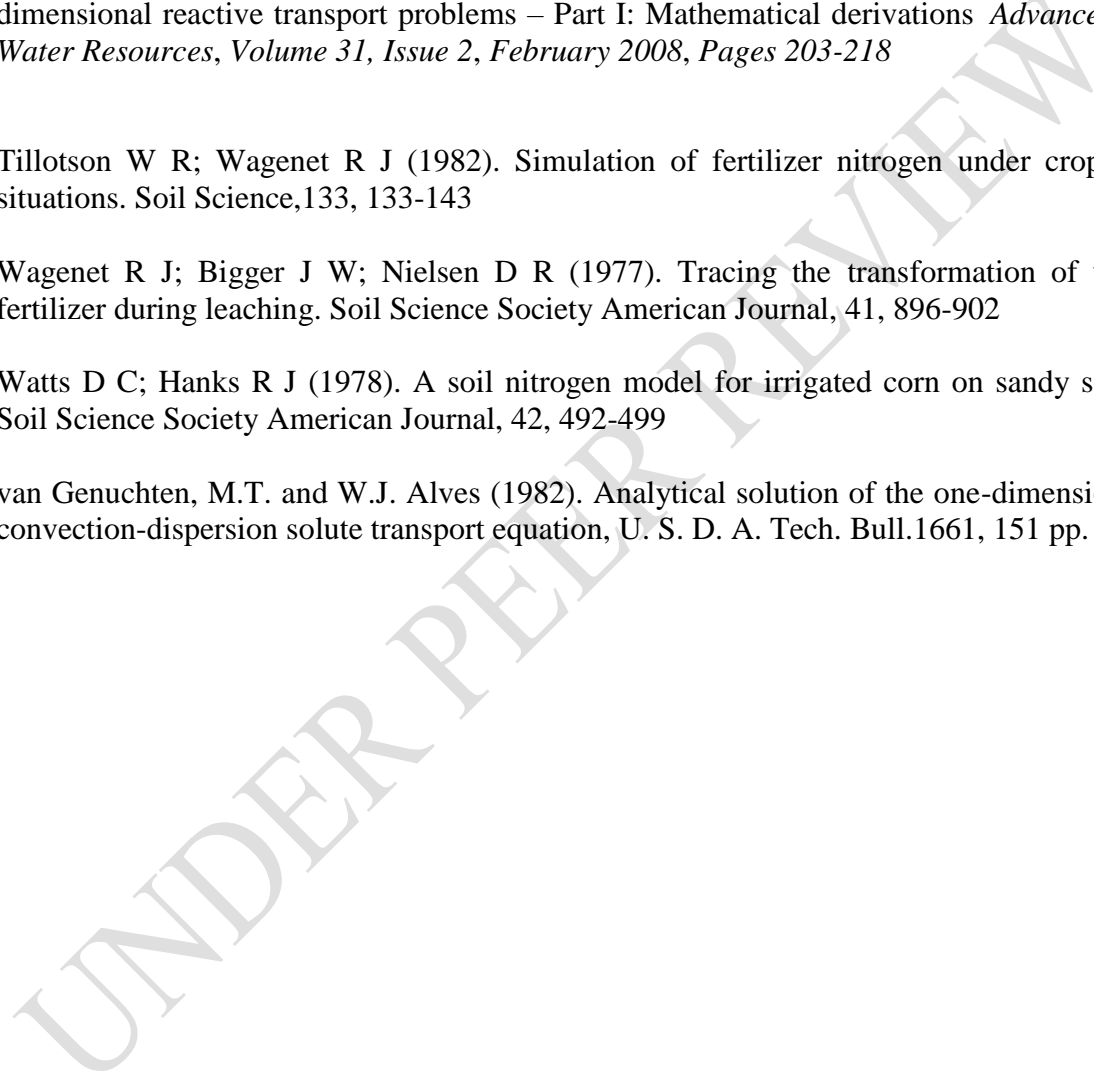
Srinivasan V., T.P. Clement (2008) Analytical solutions for sequentially coupled one-dimensional reactive transport problems – Part I: Mathematical derivations *Advances in Water Resources, Volume 31, Issue 2, February 2008, Pages 203-218*

Tillotson W R; Wagenet R J (1982). Simulation of fertilizer nitrogen under cropped situations. *Soil Science*,133, 133-143

Wagenet R J; Bigger J W; Nielsen D R (1977). Tracing the transformation of urea fertilizer during leaching. *Soil Science Society American Journal*, 41, 896-902

Watts D C; Hanks R J (1978). A soil nitrogen model for irrigated corn on sandy soils. *Soil Science Society American Journal*, 42, 492-499

van Genuchten, M.T. and W.J. Alves (1982). Analytical solution of the one-dimensional convection-dispersion solute transport equation, U. S. D. A. Tech. Bull.1661, 151 pp.



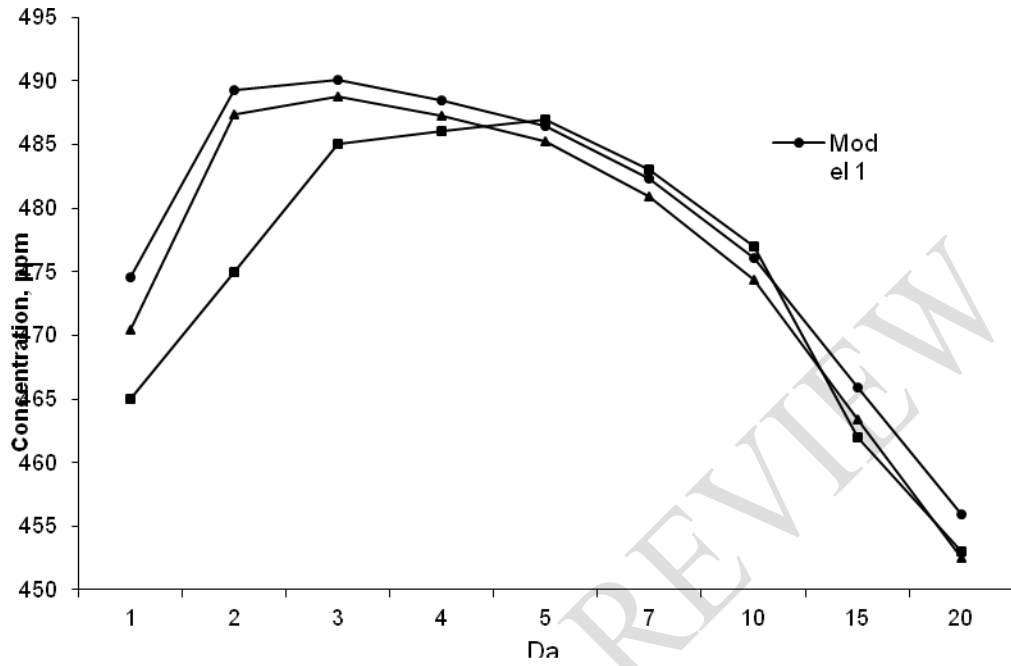


Fig.1: Performance of developed models at different days at 15 cm depth

345

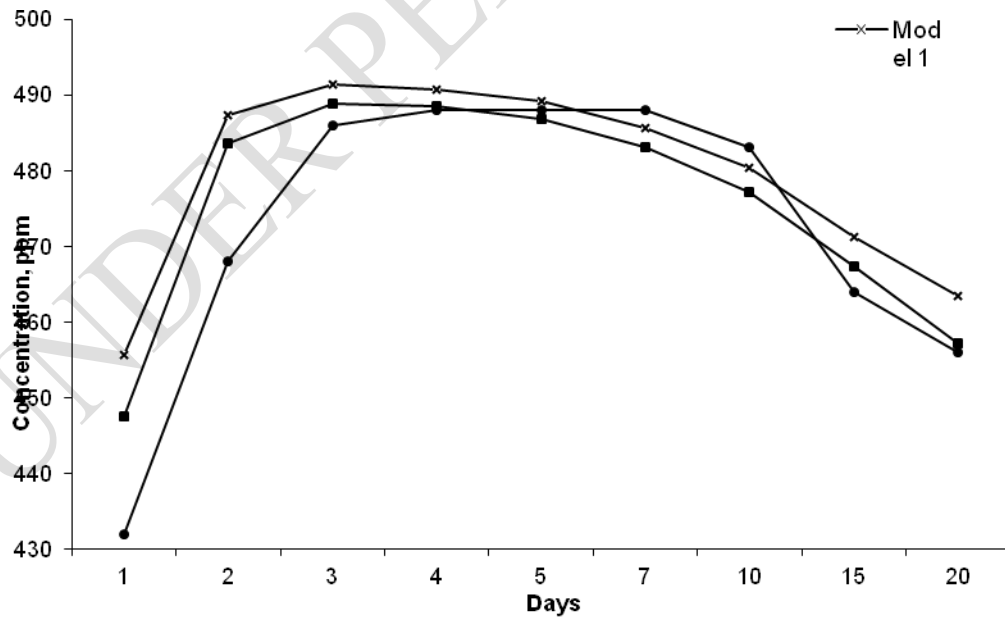


Fig.2: Performance of developed models at different days at 30 cm depth

346

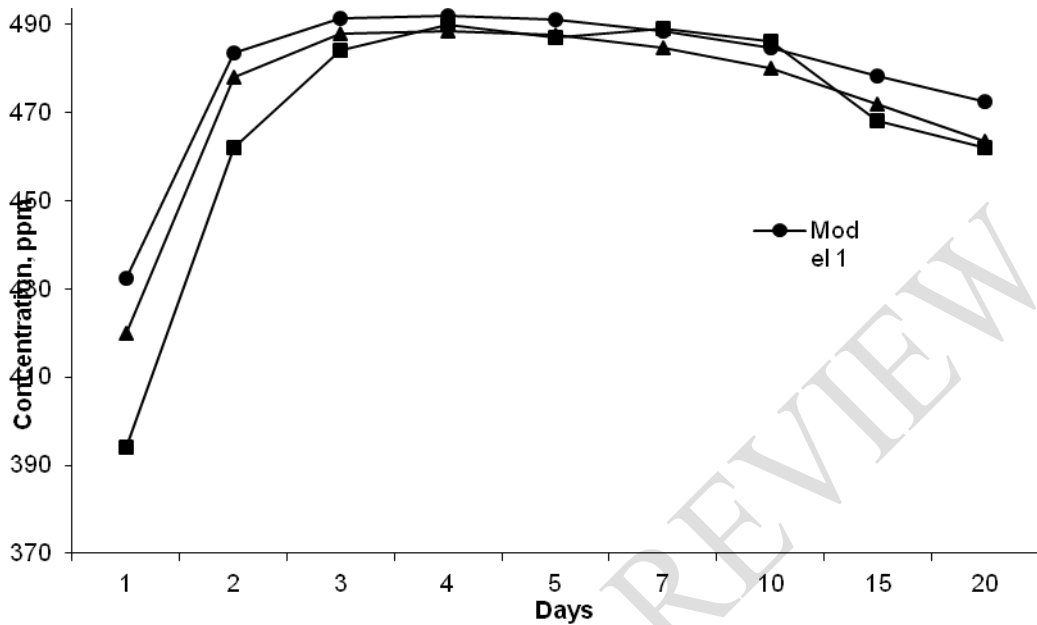


Fig.3: Performance of developed models at different days at 50 cm depth

347

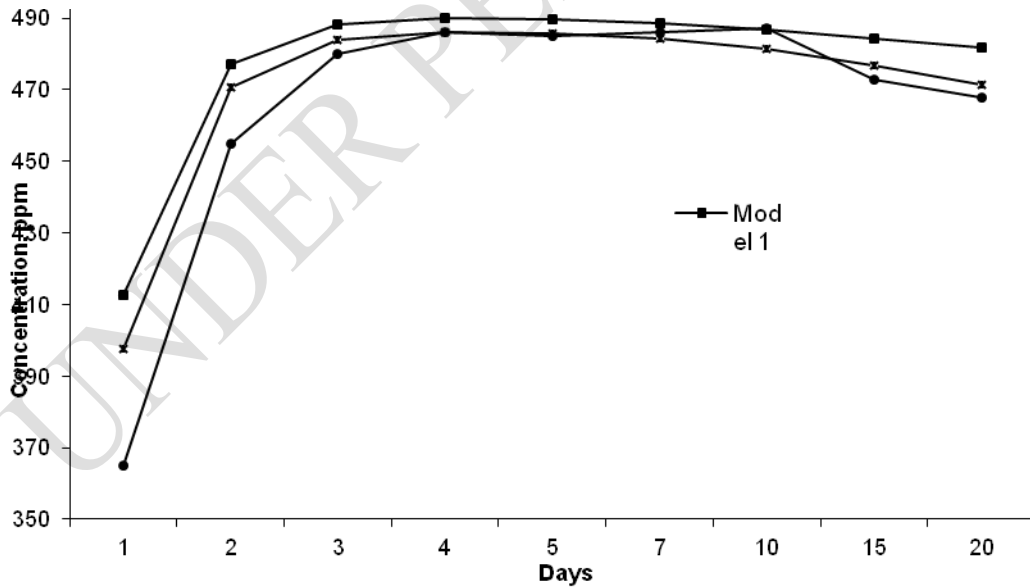


Fig.4: Performance of developed models at different days at 75 cm depth

348

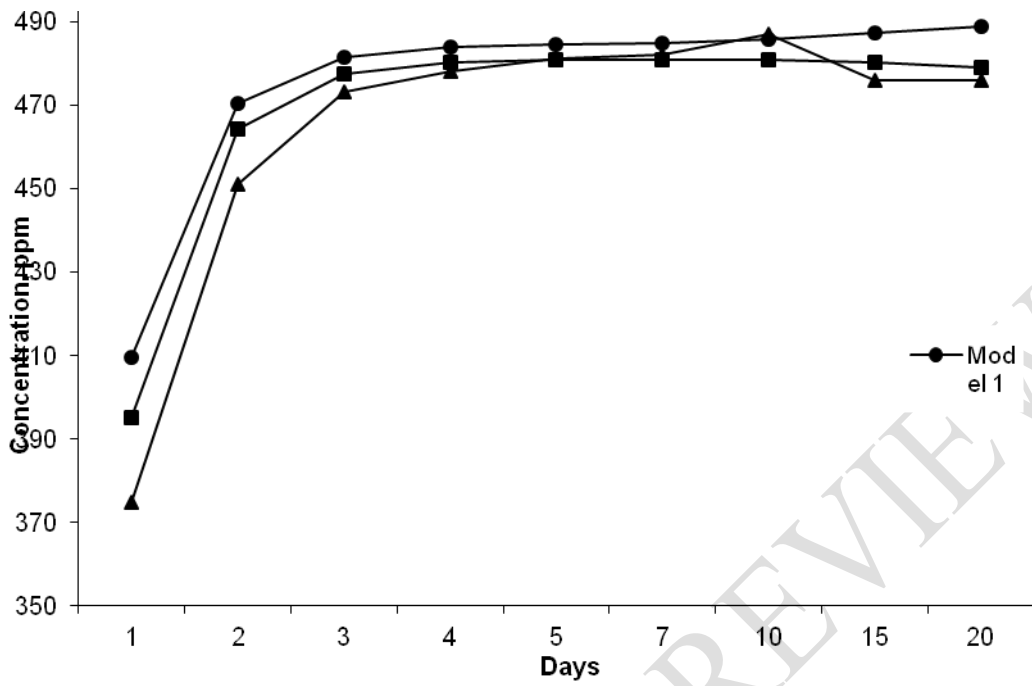


Fig.5: Performance of developed models at different days at 100 cm depth

349

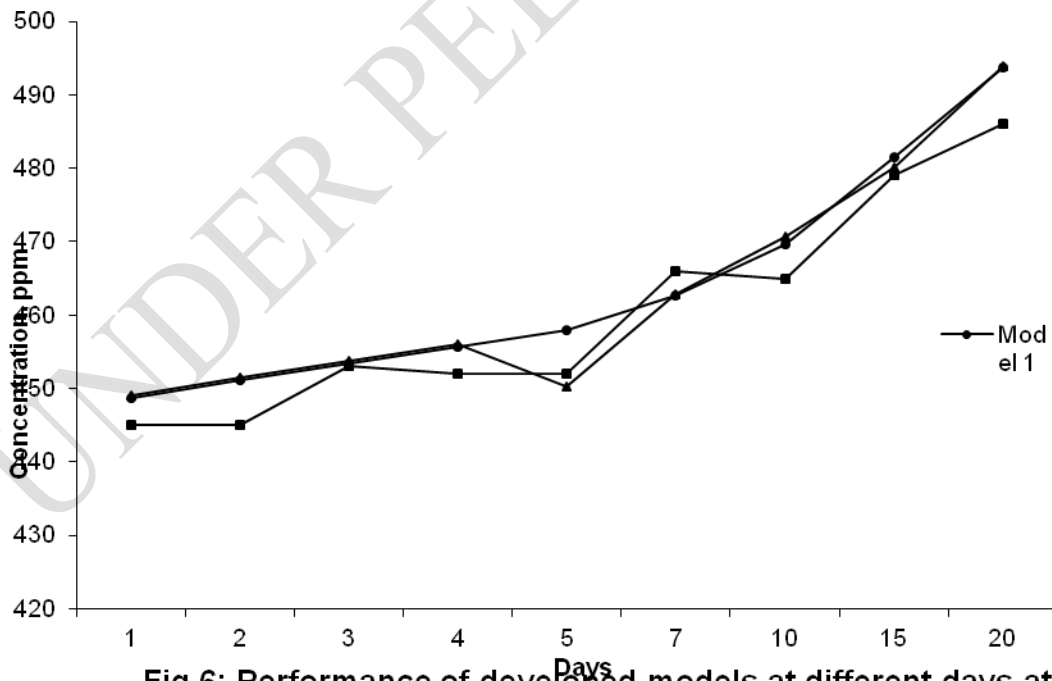


Fig.6: Performance of developed models at different days at 150 cm depth

350

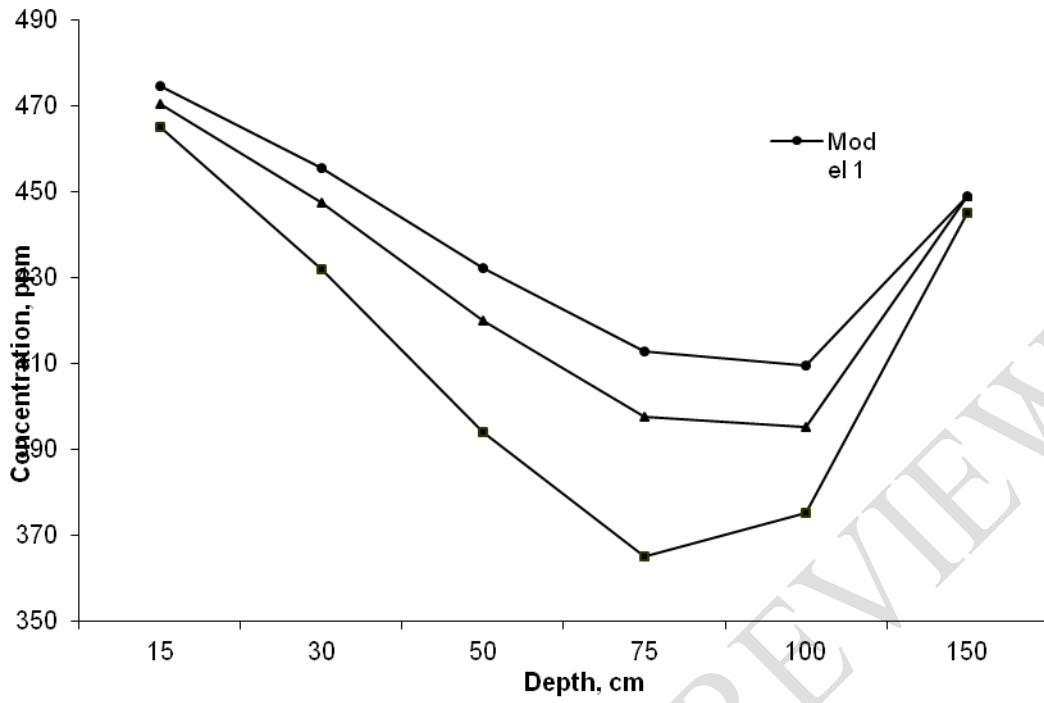


Fig. 7: Performance of developed models at different depth on first day

351

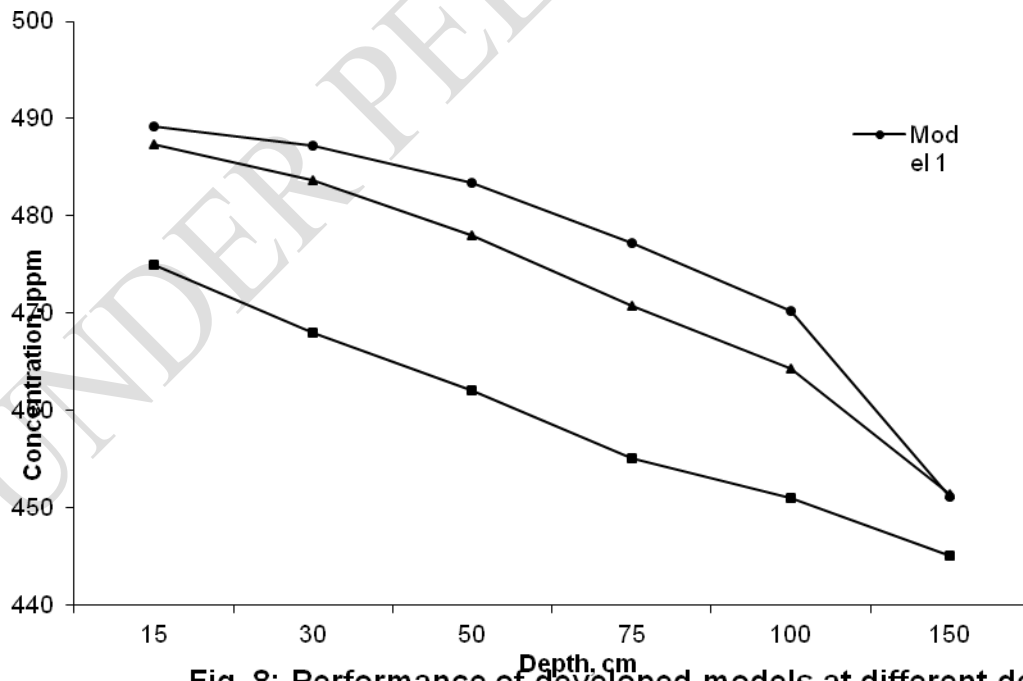


Fig. 8: Performance of developed models at different depth on second day

352

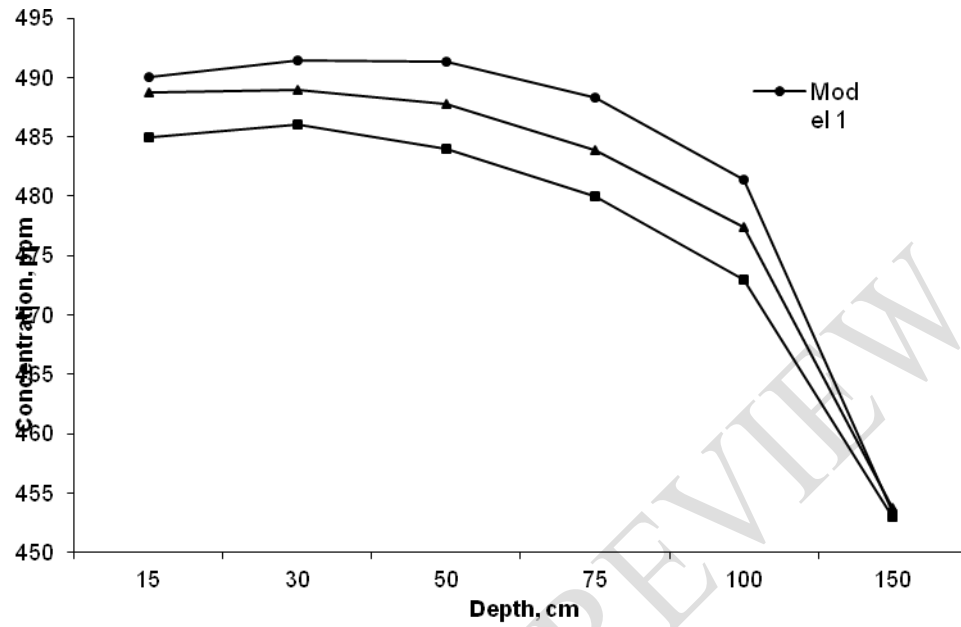


Fig.9: Performance of developed models at different depth on third day

353

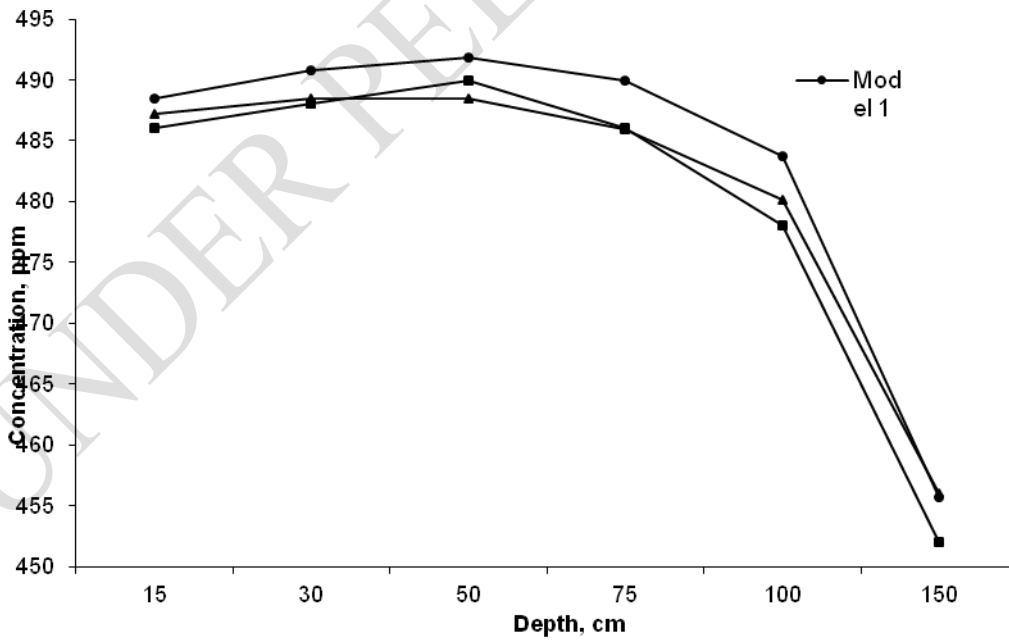


Fig. 10: Performance of developed models at different depth on fourth day

354

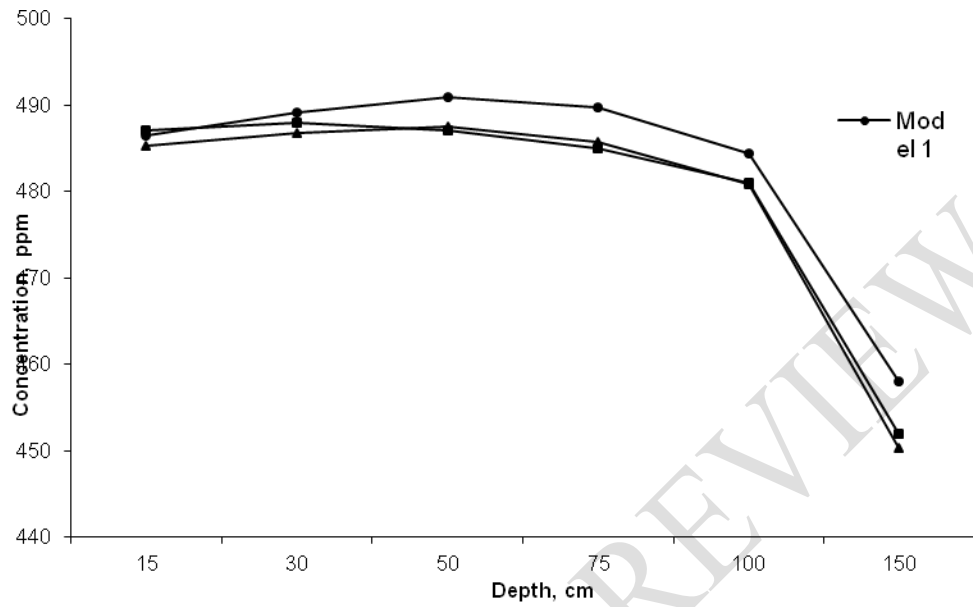


Fig. 11: Performance of developed models at different depth on fifth day

355

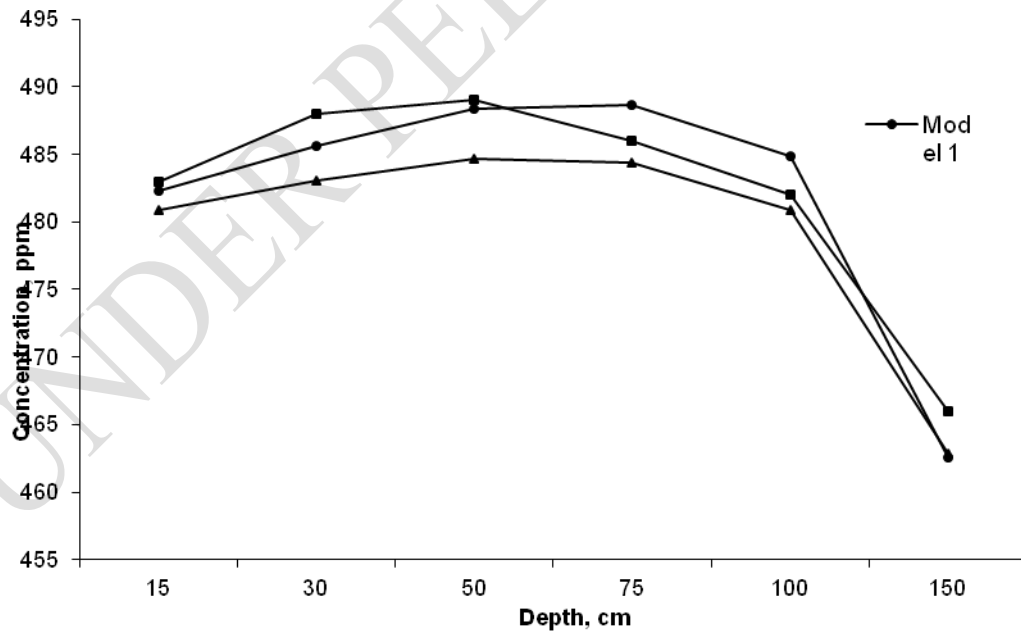


Fig. 12: Performance of developed models at different depth on seventh day

356

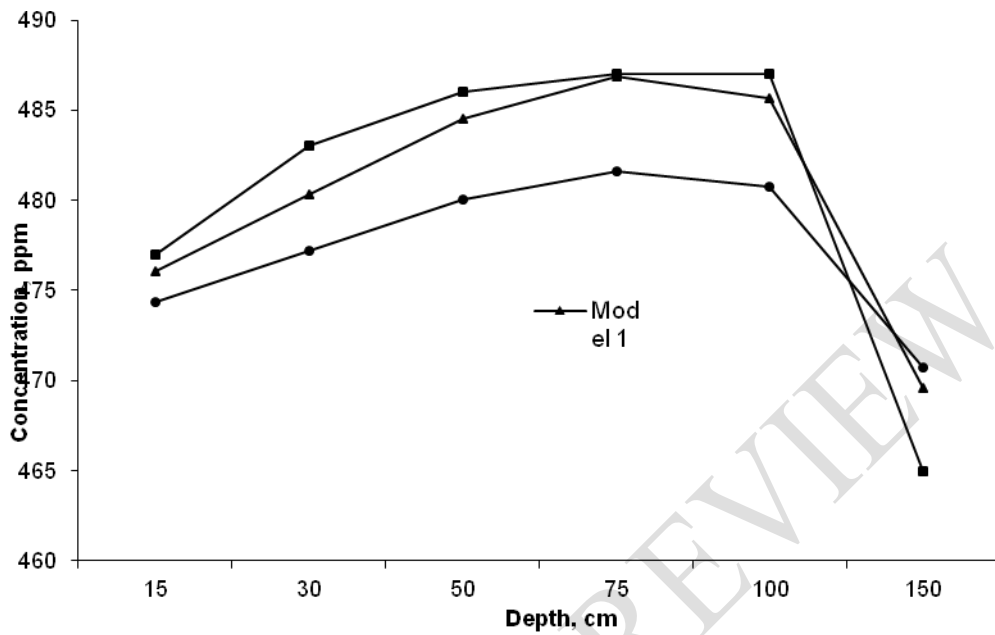


Fig. 13: Performance of developed models at different depth on tenth day

357

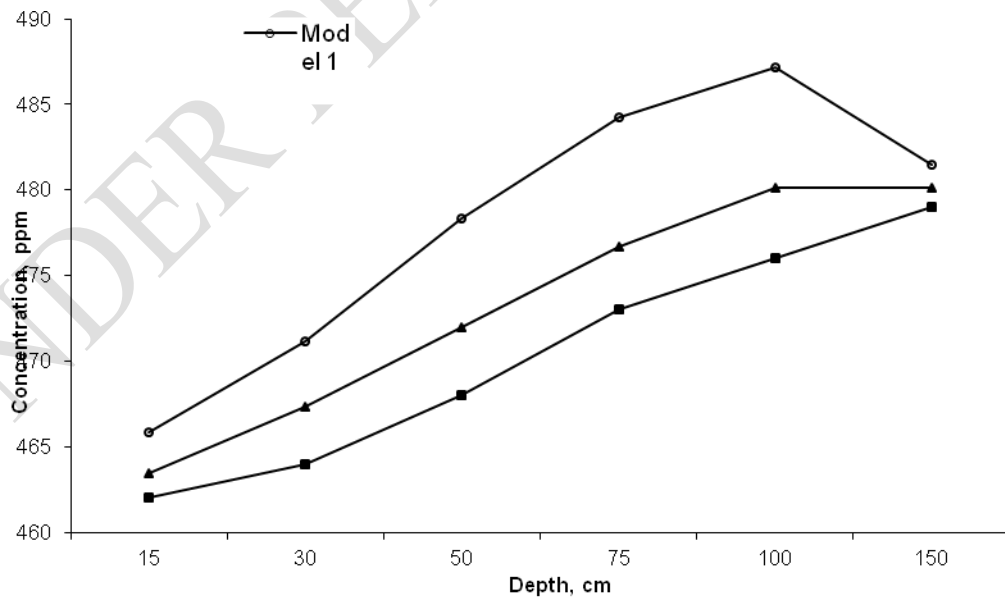


Fig. 14: Performance of developed models at different depth on fifteenth day

358

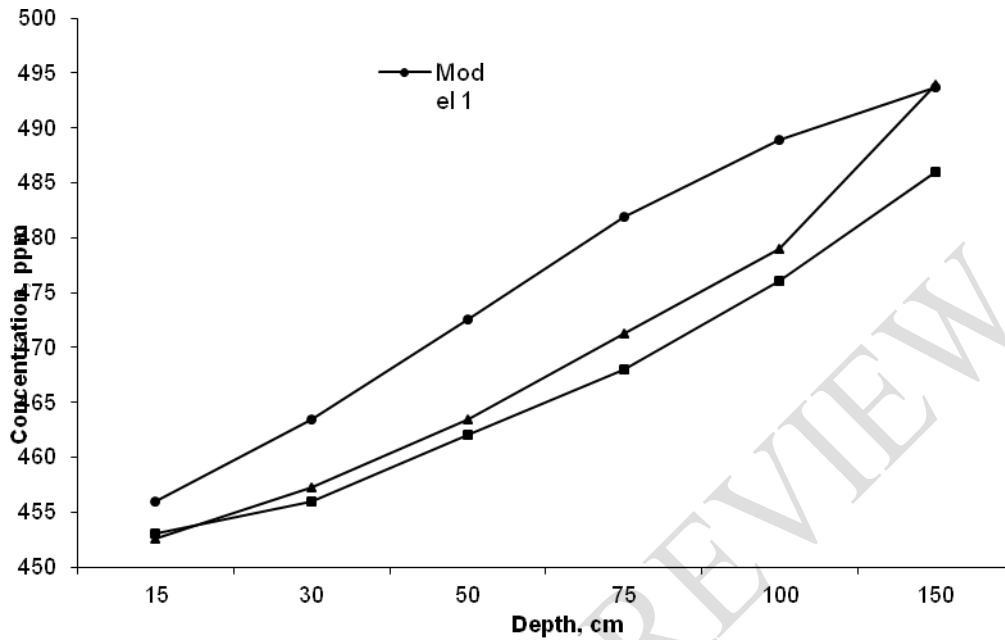


Fig.15: Performance of developed models at different depth on twelfth day

359
360
361
362

at .15 meter depth

	1	2	obser	one	model 1	model2	observed	
364								
365	1	474.56	470.43	465	15	474.56	470.43	465
366	2	489.23	487.33	475	30	455.6	447.55	432
367	3	490.07	488.733	485	50	432.33	419.96	394
368	4	488.51	487.25	486	75	412.81	397.56	365
369	5	486.49	485.21	487	100	409.46	395.2	375
370	7	482.29	480.88	483	150	448.77	449	445

371 10 476.04 474.34 477 two

372	15	465.86	463.44	462	15	489.23	487.33	475
373	20	455.96	452.55	453	30	487.26	483.66	468
374					50	483.35	477.97	462
375					75	477.22	470.71	455

at .30 meter depth

	1	5	obser	three				
376								
377	1	455.6	447.55	432	100	470.26	464.23	451
378	2	487.26	483.66	468	150	451.15	451.41	445
379	3	491.4	488.92	486	15	490.07	488.733	485
380	4	490.74	488.43	488	30	491.4	488.92	486
381	5	489.16	486.81	488	50	491.37	487.75	484
382	7	485.63	483.04	488	75	488.28	483.93	480
383	10	480.34	477.22	483	100	481.4	477.44	473

384	15	471.16	467.33	464	150	453.44	453.69	453
385	20	463.47	457.22	456				
386					four			
387					15	488.51	487.25	486
388	at .5 meter depth					30	490.74	488.43 488
389		1	5	obser	50	491.86	488.52	490
390	1	432.33	419.96	394	75	489.92	485.99	486
391	2	483.35	477.97	462	100	483.73	480.1	478
392	3	491.37	487.75	484	150	455.71	455.98	452
393	4	491.86	488.52	490				
394	5	490.89	487.5	487	five			
395	7	488.36	484.63	489	15	486.49	485.21	487
396	10	484.52	480.02	486	30	489.16	486.81	488
397	15	478.37	471.97	468	50	490.89	487.5	487
398	20	472.5	463.46	462	75	489.74	485.76	485
399					100	484.34	480.76	481
400	at .75 meter depth					150	458	450.26 452
401								
402		1	5	obser	seven			
403	1	412.81	397.56	365	15	482.29	480.88	483
404	2	477.22	470.71	455	30	485.63	483.04	488
405	3	488.28	483.93	480	50	488.36	484.63	489
406	4	489.92	485.99	486	75	488.64	484.35	486
407	5	489.74	485.76	485	100	484.84	480.83	482
408	7	488.64	484.35	486	150	462.6	462.87	466
409	10	486.91	481.61	487				
410	15	484.27	476.72	473	ten			
411	20	481.93	471.24	468	15	476.04	474.34	477
412					30	480.34	477.22	483
413	at one meter depth					50	484.52	480.02 486
414		1	5	obser	75	486.91	481.61	487
415	1	409.46	395.2	375	100	485.64	480.73	487
416	2	470.26	464.23	451	150	469.59	470.72	465
417	3	481.4	477.44	473				
418	4	483.73	480.1	478	fifteen			
419	5	484.34	480.76	481	15	465.86	463.44	462
420	7	484.84	480.83	482	30	471.16	467.33	464
421	10	485.64	480.73	487	50	478.37	471.97	468
422	15	487.13	480.11	476	75	484.27	476.72	473
423	20	488.92	478.96	476	100	487.13	480.11	476
424					150	481.48	480.11	479
425								
426	at 1.5 meter depth					twenty		
427		1	5	obser	15	455.96	452.55	453
428	1	448.77	449	445	30	463.47	457.22	456
429	2	451.15	451.41	445	50	472.5	463.46	462

430	3	453.44	453.69	453	75	481.93	471.24	468
431	4	455.71	455.98	452	100	488.92	478.96	476
432	5	458	450.26	452	150	493.67	493.95	486
433	7	462.6	462.87	466				
434	10	469.59	470.72	465				
435	15	481.48	480.11	479				
436	20	493.67	493.95	486				

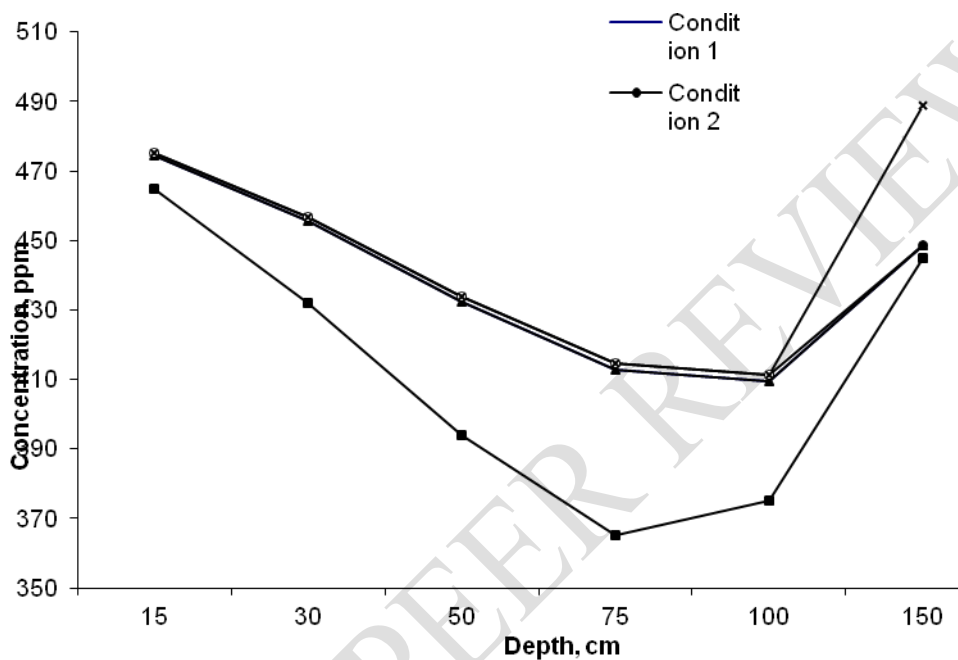


Fig. 16: Performance of model-1 under limiting conditions at different depth on first day

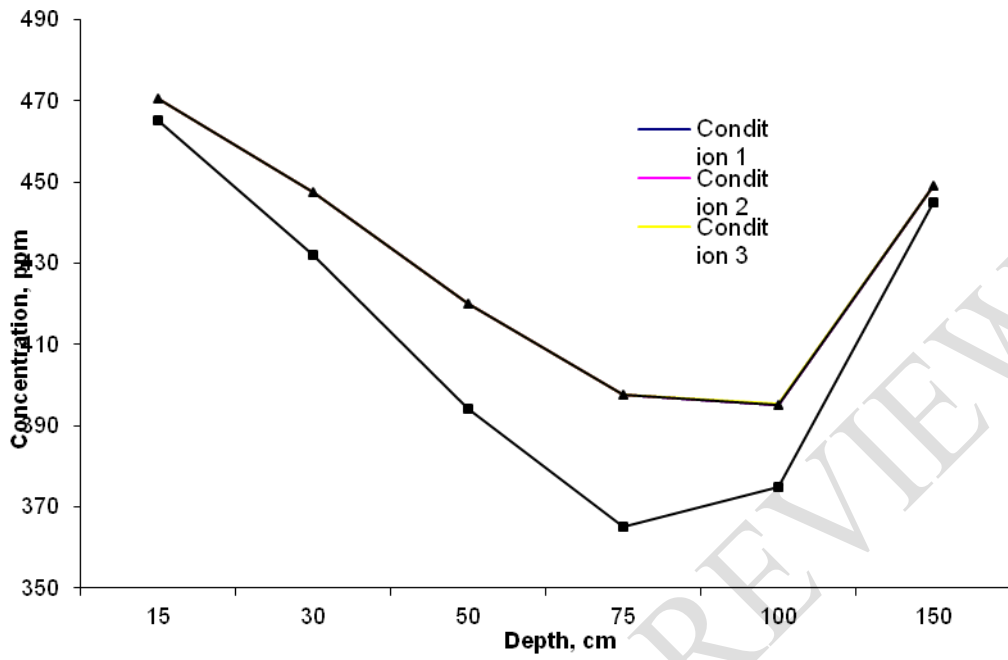


Fig. 17: Performance of model-2 under limiting conditions at different depth on first day