# Properties of Generalized 6-primes Numbers

Yüksel Soykan

Department of Mathematics, Art and Science
Faculty, Zonguldak Bülent Ecevit University, 67100,
Zonguldak, Turkey

**Abstract.** In this paper, we introduce the generalized 6-primes sequence and we deal with, in detail, three special cases which we call them 6-primes, Lucas 6-primes and modified 6-primes sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

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## 1. Introduction

In this paper, we investigate the generalized 6-primes sequences and we investigate, in detail, three special cases which we call them 6-primes, Lucas-6-primes and modified 6-primes sequences.

The sequence of Fibonacci numbers $\{F_n\}$ and the sequence of Lucas numbers $\{L_n\}$ are defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \quad F_0 = 0, \; F_1 = 1,$$

and

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \quad L_0 = 2, \; L_1 = 1$$

respectively. The generalizations of Fibonacci and Lucas sequences produce several nice and interesting sequences.

The generalized Hexanacci sequence $\{W_n(W_0, W_1, W_2, W_3, W_4, W_5; r_1, r_2, r_3, r_4, r_5, r_6)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined by the sixth-order recurrence relations

$$\text{(1.1)} \qquad \begin{aligned} W_n &= r_1W_{n-1} + r_2W_{n-2} + r_3W_{n-3} + r_4W_{n-4} + r_5W_{n-5} + r_6W_{n-6}, \\ W_0 &= c_0, W_1 = c_1, W_2 = c_2, \; W_3 = c_3, W_4 = c_4, W_5 = c_5, \; n \geq 6 \end{aligned}$$

where $W_0, W_1, W_2, W_3, W_4, W_5$ are arbitrary real or complex numbers and $r_1, r_2, r_3, r_4, r_5,$

$r_6$ are real numbers. The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r_5}{r_6} W_{-n+1} - \frac{r_4}{r_6} W_{-n+2} - \frac{r_3}{r_6} W_{-n+3} - \frac{r_2}{r_6} W_{-n+4} - \frac{r_1}{r_6} W_{-n+5} + \frac{1}{r_6} W_{-n+6}$$

for $n = 1, 2, 3, \ldots$ when $r_6 \neq 0$. Therefore, recurrence (1.1) holds for all integer $n$. Hexanacci sequence has been studied by many authors, see for example [4], [5].

As $\{W_n\}$ is a sith order recurrence sequence (di¤erence equation), it's characteristic equation is

$$x^6 \quad r_1 x^5 \quad r_2 x^4 \quad r_3 x^3 \quad r_4 x^2 \quad r_5 x \quad r_6 = 0 \qquad (1.2)$$

whose roots are $1;\ 2;\ 3;\ 4;\ 5;\ 6$: Generalized Hexanacci numbers can be expressed, for all integers $n$; using

Binet's formula.

Theorem 1. (Binet formula of generalized Hexanacci numbers)

$$W_n = \sum_{k=1}^{6} \frac{b_k\ {}^n_k}{\prod_{j=1,\,k\neq j}^{6} (\ _k\ \ _j)} \qquad (1.3)$$

where

$b_1$ = $W_5$ ( 2 + 3 + 4 + 5 + 6 )$W_4$ + ( 2 5 + 2 3 + 2 6 + 5 3 + 5 6 + 2
( 2 5 3 + 2 5 6 + 2 5 4 + 2 3 6 + 5 3 6 + 2 3 4 + 2 6 4 +
+( 2 5 3 6 + 2 5 3 4 + 2 5 6 4 + 2 3 6 4 + 5 3 6 4 )$W_1$ 2

$b_2$ = $W_5$ ( 1 + 3 + 4 + 5 + 6 )$W_4$ + ( 1 5 + 1 3 + 1 6 + 1 4 + 5 3 + 5
( 1 5 3 + 1 5 6 + 1 5 4 + 1 3 6 + 1 3 4 + 1 6 4 + 5 3 6 +
+( 1 5 3 6 + 1 5 3 4 + 1 5 6 4 + 1 3 6 4 + 5 3 6 4 )$W_1$ 1

$b_3$ = $W_5$ ( 1 + 2 + 4 + 5 + 6 )$W_4$ + ( 1 2 + 1 5 + 1 6 + 2 5 + 1 4 + 2
( 1 2 5 + 1 2 6 + 1 5 6 + 1 2 4 + 1 5 4 + 2 5 6 + 1 6 4 +
+( 1 2 5 6 + 1 2 5 4 + 1 2 6 4 + 1 5 6 4 + 2 5 6 4 )$W_1$ 1

$b_4$ = $W_5$ ( 1 + 2 + 3 + 5 + 6 )$W_4$ + ( 1 2 + 1 5 + 1 3 + 1 6 + 2 5 + 2
( 1 2 5 + 1 2 3 + 1 2 6 + 1 5 3 + 1 5 6 + 1 3 6 + 2 5 3 +
+( 1 2 5 3 + 1 2 5 6 + 1 2 3 6 + 1 5 3 6 + 2 5 3 6 )$W_1$ 1

$b_5$ = $W_5$ ( 1 + 2 + 3 + 4 + 6 )$W_4$ + ( 1 2 + 1 3 + 1 6 + 1 4 + 2 3 + 2
( 1 2 3 + 1 2 6 + 1 2 4 + 1 3 6 + 1 3 4 + 1 6 4 + 2 3 6 +
+( 1 2 3 6 + 1 2 3 4 + 1 2 6 4 + 1 3 6 4 + 2 3 6 4 )$W_1$ 1

$b_6$ = $W_5$ ( 1 + 2 + 3 + 4 + 5 )$W_4$ + ( 1 2 + 1 5 + 1 3 + 2 5 + 1 4 + 2
( 1 2 5 + 1 2 3 + 1 5 3 + 1 2 4 + 1 5 4 + 2 5 3 + 1 3 4 +
+( 1 2 5 3 + 1 2 5 4 + 1 2 3 4 + 1 5 3 4 + 2 5 3 4 )$W_1$ 1

Usually, it is customary to choose $r_1; r_2; r_3; r_4; r_5; r_6$ so that the Equ. (1.2) has at least one real (say $1$) solutions. Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers $n$; for a proof of this result see [1]:This result of Howard and Saidak [1] is even true in the case of higher-order recurrence relations.

In this paper we consider the case $r_1 = 2; r_2 = 3; r_3 = 5; r_4 = 7; r_5 = 11; r_6 = 13$ and in this case we write

$V_n = W_n$: For recent relevant studies, see [9,10,11,12,13]. Note that $2; 3; 5; 7; 11$ and $13$ are prime numbers. Prime numbers are numbers that have only 2 factors: 1 and themselves. For more details on prime numbers, see for example [3].

A generalized 6-primes sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0; V_1; V_2; V_3; V_4; V_5)\}_{n\geq 0}$ is defined by the sixth-order recurrence relations

$$V_n = 2V_{n-1} + 3V_{n-2} + 5V_{n-3} + 7V_{n-4} + 11V_{n-5} + 13V_{n-6} \quad (1.4)$$

with the initial values $V_0 = c_0; V_1 = c_1; V_2 = c_2; V_3 = c_3; V_4 = c_4; V_5 = c_5$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{11}{13}V_{-(n-1)} - \frac{7}{13}V_{-(n-2)} - \frac{5}{13}V_{-(n-3)} - \frac{3}{13}V_{-(n-4)} - \frac{2}{13}V_{-(n-5)} + \frac{1}{13}V_{-(n-6)}$$

for $n = 1; 2; 3; \ldots$. Therefore, recurrence (1.4) holds for all integer $n$.

(1.3) can be used to obtain Binet formula of generalized 6-primes numbers. Binet formula of generalized 6-primes numbers can be given as

$$V_n = \sum_{k=1}^{6} \frac{b_k \alpha_k^n}{\prod_{j=1, k\neq j}^{6} (\alpha_k - \alpha_j)} \quad (1.5)$$

where

$b_1 = V_5 - (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)V_4 + (\alpha_2\alpha_5 + \alpha_2\alpha_3 + \alpha_2\alpha_6 + \alpha_5\alpha_3 + \alpha_5\alpha_6 + \alpha_2$ [illegible]

$-(\alpha_2\alpha_5\alpha_3 + \alpha_2\alpha_5\alpha_6 + \alpha_2\alpha_5\alpha_4 + \alpha_2\alpha_3\alpha_6 + \alpha_5\alpha_3\alpha_6 + \alpha_2\alpha_3\alpha_4 + \alpha_2\alpha_6\alpha_4 + \alpha_5$ [illegible]

$+(\alpha_2\alpha_5\alpha_3\alpha_6 + \alpha_2\alpha_5\alpha_3\alpha_4 + \alpha_2\alpha_5\alpha_6\alpha_4 + \alpha_2\alpha_3\alpha_6\alpha_4 + \alpha_5\alpha_3\alpha_6\alpha_4)V_1 - \alpha_2\alpha_5$ [illegible]

$b_2 = V_5 - (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)V_4 + (\alpha_1\alpha_5 + \alpha_1\alpha_3 + \alpha_1\alpha_6 + \alpha_1\alpha_4 + \alpha_5\alpha_3 + \alpha_5$ [illegible]

$-(\alpha_1\alpha_5\alpha_3 + \alpha_1\alpha_5\alpha_6 + \alpha_1\alpha_5\alpha_4 + \alpha_1\alpha_3\alpha_6 + \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_6\alpha_4 + \alpha_5\alpha_3\alpha_6 + \alpha_5$ [illegible]

$+(\alpha_1\alpha_5\alpha_3\alpha_6 + \alpha_1\alpha_5\alpha_3\alpha_4 + \alpha_1\alpha_5\alpha_6\alpha_4 + \alpha_1\alpha_3\alpha_6\alpha_4 + \alpha_5\alpha_3\alpha_6\alpha_4)V_1 - \alpha_1\alpha_5$ [illegible]

$b_3 = V_5 - (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6)V_4 + (\alpha_1\alpha_2 + \alpha_1\alpha_5 + \alpha_1\alpha_6 + \alpha_2\alpha_5 + \alpha_1\alpha_4 + \alpha_2$ [illegible]

$-(\alpha_1\alpha_2\alpha_5 + \alpha_1\alpha_2\alpha_6 + \alpha_1\alpha_5\alpha_6 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_5\alpha_4 + \alpha_2\alpha_5\alpha_6 + \alpha_1\alpha_6\alpha_4 + \alpha_2$ [illegible]

$+(\alpha_1\alpha_2\alpha_5\alpha_6 + \alpha_1\alpha_2\alpha_5\alpha_4 + \alpha_1\alpha_2\alpha_6\alpha_4 + \alpha_1\alpha_5\alpha_6\alpha_4 + \alpha_2\alpha_5\alpha_6\alpha_4)V_1 - \alpha_1\alpha_2$ [illegible]

$b_4 = V_5 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6)V_4 + (\alpha_1\alpha_2 + \alpha_1\alpha_5 + \alpha_1\alpha_3 + \alpha_1\alpha_6 + \alpha_2\alpha_5 + \alpha_2$ [illegible]

$-(\alpha_1\alpha_2\alpha_5 + \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_6 + \alpha_1\alpha_5\alpha_3 + \alpha_1\alpha_5\alpha_6 + \alpha_1\alpha_3\alpha_6 + \alpha_2\alpha_5\alpha_3 + \alpha_2$ [illegible]

$+(\alpha_1\alpha_2\alpha_5\alpha_3 + \alpha_1\alpha_2\alpha_5\alpha_6 + \alpha_1\alpha_2\alpha_3\alpha_6 + \alpha_1\alpha_5\alpha_3\alpha_6 + \alpha_2\alpha_5\alpha_3\alpha_6)V_1 - \alpha_1\alpha_2$ [illegible]

$b_5 = V_5 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6)V_4 + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_6 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2$ [illegible]

$-(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_6 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_6 + \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_6\alpha_4 + \alpha_2\alpha_3\alpha_6 + \alpha_2$ [illegible]

$+(\alpha_1\alpha_2\alpha_3\alpha_6 + \alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_6\alpha_4 + \alpha_1\alpha_3\alpha_6\alpha_4 + \alpha_2\alpha_3\alpha_6\alpha_4)V_1 - \alpha_1\alpha_2$ [illegible]

$b_6 = V_5 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)V_4 + (\alpha_1\alpha_2 + \alpha_1\alpha_5 + \alpha_1\alpha_3 + \alpha_2\alpha_5 + \alpha_1\alpha_4 + \alpha_2$ [illegible]

$-(\alpha_1\alpha_2\alpha_5 + \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_5\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_5\alpha_4 + \alpha_2\alpha_5\alpha_3 + \alpha_1\alpha_3\alpha_4 + \alpha_2$ [illegible]

+( 1 2 5 3 + 1 2 5 4 + 1 2 3 4 + 1 5 3 4 + 2 5 3 4)V1 1 2
5 3 4)V0;

Here, $_1$; $_2$; $_3$; $_4$; $_5$ and $_6$ are the roots of the equation

$$x^6 \quad 2x^5 \quad 3x^4 \quad 5x^3 \quad 7x^2 \quad 11x \quad 13 = 0: \qquad (1.6)$$

Moreover, the approximate value of the roots $_1$; $_2$; $_3$; $_4$; $_5$ and $_6$ of Equation (1.6) are given by

$$\begin{aligned}
_1 &= 3{:}515372711921757;\\
_2 &= \quad 1{:}183212731145181;\\
_3 &= \quad 0{:}7228110394202282 + 1{:}063369120765496i;\\
_4 &= \quad 0{:}722811039420228\,2 \quad 1{:}063369120765496i;\\
_5 &= 0{:}556731049\,031939\,9 + 1{:}257207305141223i;\\
_6 &= 0{:}556731049031939\,9 \quad 1{:}257207305141223i;
\end{aligned}$$

The first few generalized 6-primes numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized 6-primes numbers

| $n$ | $V_n$ | $V$ |
|---|---|---|
| 0 | $V_0$ | |
| 1 | $V_1$ | $\frac{1}{13}V_5 \quad \frac{7}{13}V_1 \quad \frac{5}{13}V_2 \quad \frac{3}{13}V_3 \quad \frac{2}{13}V_4 \quad \frac{11}{13}V_0$ |
| 2 | $V_2$ | $\frac{30}{169}V_0 + \frac{1}{169}V_1 + \frac{7}{169}V_2 + \frac{3}{169}V_3 + \frac{11}{169}V_4 + \frac{1}{169}V_5$ |
| 3 | $V_3$ | $\frac{365}{2197}\ \ -\frac{2}{2197}V_3 \quad \frac{59}{2197}V_1 \quad \frac{17}{2197}V_2 \quad \frac{20}{2197}V_2 \quad \frac{30}{2197}V_0 \quad 2197\,V_4 + 2197\,V_5$ |
| 4 | $V_4$ | $\frac{1888}{28\,561}\ \ -\frac{451}{28\,561}V_0 + \frac{561}{28\,561}V_1 + \frac{211}{28\,561}V_2 \quad \frac{738}{28\,561}V_2 \quad \frac{174}{28\,561}V_3 + 28\,561\,V_4 \quad 28\,561\,V_5$ |
| 5 | $V_5$ | $\frac{779}{371\,293} \quad \frac{59}{371\,293}V_1 \quad \frac{905}{371\,293}V_1 \quad \frac{961}{371\,293}V_0 \quad \frac{393}{371\,293} \quad \frac{603}{371\,293}V_2 + \frac{1888}{371\,293}V_3 \quad 371\,293\,V_4 + 371\,293\,V_5$ (14, 36) |
| 6 | $13V_0 + 11V_1 + 7V_2 + 5V_3 + 3V_4 + 2V_5$ | $\frac{941\,082}{4826\,809}V_0 \quad \frac{376}{4826\,809}\ \frac{158}{4826\,809}V_1 + \frac{125}{4826\,809}\ \frac{615}{4826\,809}V_2 \quad \frac{33}{4826\,809}\ \frac{779}{4826\,809}V_3 + \frac{54}{4826\,809}\ \frac{354}{4826\,809}V_4 \quad \frac{14}{4826\,809}\ \frac{905}{4826\,809}V_5$ |

Now we define three special cases of the sequence $\{V_n\}$. 6-primes sequence $\{G_n\}_{n\geq 0}$, Lucas 6-primes sequence $\{H_n\}_{n\geq 0}$ and modified 6-primes sequence $\{E_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_{n+6} = 2G_{n+5} + 3G_{n+4} + 5G_{n+3} + 7G_{n+2} + 11G_{n+1} + 13G_n; \quad G_0 = 0; G_1 = 0; G_2 = 0; G_3 = 0; G_4 = 1; G_5 = 2;$$

$$H_{n+6} = 2H_{n+5} + 3H_{n+4} + 5H_{n+3} + 7H_{n+2} + 11H_{n+1} + 13H_n; \quad H_0 = 6; H_1 = 2; H_2 = 10; H_3 = 41; H_4 = 150; H_5 = 542;$$

and

$$E_{n+6} = 2E_{n+5} + 3E_{n+4} + 5E_{n+3} + 7E_{n+2} + 11E_{n+1} + 13E_n; \quad E_0 = 0; E_1 = 0; E_2$$

$= 0, E_3 = 0, E_4 = 1, E_5 = 1.$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{11}{13}G_{-(n-1)} - \frac{7}{13}G_{-(n-2)} - \frac{5}{13}G_{-(n-3)} - \frac{3}{13}G_{-(n-4)} - \frac{2}{13}G_{-(n-5)} + \frac{1}{13}G_{-(n-6)}, \tag{1.7}$$

$$H_{-n} = -\frac{11}{13}H_{-(n-1)} - \frac{7}{13}H_{-(n-2)} - \frac{5}{13}H_{-(n-3)} - \frac{3}{13}H_{-(n-4)} - \frac{2}{13}H_{-(n-5)} + \frac{1}{13}H_{-(n-6)} \tag{1.8}$$

and

$$E_{-n} = -\frac{11}{13}E_{-(n-1)} - \frac{7}{13}E_{-(n-2)} - \frac{5}{13}E_{-(n-3)} - \frac{3}{13}E_{-(n-4)} - \frac{2}{13}E_{-(n-5)} + \frac{1}{13}E_{-(n-6)} \tag{1.9}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (1.7), (1.8) and (1.9) hold for all integer $n$.

Note that the sequences $G_n$, $H_n$ and $E_n$ are not indexed in [6] yet. Next, we present the first few values of the 6-primes, Lucas 6-primes and modified 6-primes numbers with positive and negative subscripts:

Table 2. The first few values of the special sisth-order numbers with positive and negative subscripts.

| | n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $G_n$ | 0 | 0 | 0 | 0 | 1 | 2 | 7 | 25 | 88 | 311 |
| 1 | $G_{-n}$ | | 0 | 13 169 | 1 1 | 30 2197 28561 | 174 | 188 8 371293 4826809 | 1490 5 | 9410 82 62748517 815730721 | 15241956 |
| | $H_n$ | 6 | 2 | 10 | 41 | 150 | 542 | 1909 | 6617 | 23302 | 81977 |
| 11 | $H_{-n}$ | 61 28561 | 13 | 371293 | 86 3 169 | 202 5 2197 | 607 56 | 4839 977 4826809 62748517 | 40 9115 74 | 10092 2415 815730721 604499373 | 1281284909 10 |
| | $E_n$ | 0 | 0 | 0 | 0 | 1 | 1 | 5 | 18 | 63 | 223 |
| 1 | $E_{-n}$ | 13 | | 2 4 169 2197 | 17 3 | 564 28561 371293 | 415 6 | 3944 9 4826809 62748517 | 1134 847 | 2747 6022 815730721 604499373 | 301397417 10 |

For all integers n; 6-primes, Lucas 6-primes and modified 6-primes numbers (using initial conditions in (1.5))

can be expressed using Binet's formulas as

$$G_n = \sum_{k=1}^{6} \frac{\alpha_k^{n+1}}{\prod_{j=1, k\neq j}^{6} (\alpha_k - \alpha_j)};$$

$$H_n = \sum_{k=1}^{6} \alpha_k^n = \alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n + \alpha_5^n + \alpha_6^n;$$

$$E_n = \sum_{k=1}^{6} \frac{(\alpha_k - 1)\alpha_k^{n+1}}{\prod_{j=1, k\neq j}^{6} (\alpha_k - \alpha_j)};$$

respectively.

## 2. Generating Functions

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence $V_n$:

Lemma 2. Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized 6-primes sequence $\{V_n\}_{n \geq 0}$: Then, $\sum_{n=0}^{\infty} V_n x^n$

is given by

$$\sum_{n=0}^{\infty} V_n x^n = \frac{}{1 - 2x - 3x^2 - 5x^3 - 7x^4 - 11x^5 - 13x^6}. \qquad (2.1)$$

where

$$
\begin{aligned}
&= V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2 + (V_3 - 2V_2 - 3V_1 - 5V_0)x^3 \\
&\quad + (V_4 - 2V_3 - 3V_2 - 5V_1 - 7V_0)x^4 + (V_5 - 2V_4 - 3V_3 - 5V_2 - 7V_1 - 11V_0)x^5 \\
&= V_0 + \sum_{i=1}^{6-1} x^i \left( V_i - \sum_{j=1}^{i} r_j V_{i-j} \right)
\end{aligned}
$$

*Proof.* Using the definition of generalized 6-primes numbers, and substracting $2x\sum_{n=0}^{\infty}V_n x^n$; $3x^2\sum_{n=0}^{\infty}V_n x^n$; $5x^3\sum_{n=0}^{\infty}V_n x^n$; $7x^4\sum_{n=0}^{\infty}V_n x^n$; $11x^5\sum_{n=0}^{\infty}V_n x^n$ and $13x^6\sum_{n=0}^{\infty}V_n x^n$ from $\sum_{n=0}^{\infty}V_n x^n$ we obtain

$$
\begin{aligned}
&(1-2x-3x^2-5x^3-7x^4-11x^5-13x^6)\sum_{n=0}^{\infty}V_n x^n \\
&= \sum_{n=0}^{\infty}V_n x^n - 2x\sum_{n=0}^{\infty}V_n x^n - 3x^2\sum_{n=0}^{\infty}V_n x^n - 5x^3\sum_{n=0}^{\infty}V_n x^n - 7x^4\sum_{n=0}^{\infty}V_n x^n \\
&\quad - 11x^5\sum_{n=0}^{\infty}V_n x^n - 13x^6\sum_{n=0}^{\infty}V_n x^n \\
&= \sum_{n=0}^{\infty}V_n x^n - 2\sum_{n=0}^{\infty}V_n x^{n+1} - 3\sum_{n=0}^{\infty}V_n x^{n+2} - 5\sum_{n=0}^{\infty}V_n x^{n+3} - 7\sum_{n=0}^{\infty}V_n x^{n+4} \\
&\quad - 11\sum_{n=0}^{\infty}V_n x^{n+5} - 13\sum_{n=0}^{\infty}V_n x^{n+6} \\
&= \sum_{n=0}^{\infty}V_n x^n - 2\sum_{n=1}^{\infty}V_{n-1} x^n - 3\sum_{n=2}^{\infty}V_{n-2} x^n - 5\sum_{n=3}^{\infty}V_{n-3} x^n - 7\sum_{n=4}^{\infty}V_{n-4} x^n \\
&\quad - 11\sum_{n=5}^{\infty}V_{n-5} x^n - 13\sum_{n=6}^{\infty}V_{n-6} x^n \\
&= (V_0+V_1x+V_2x^2+V_3x^3+V_4x^4+V_5x^5) - 2(V_0x+V_1x^2+V_2x^3+V_3x^4+V_4x^5) \\
&\quad - 3(V_0x^2+V_1x^3+V_2x^4+V_3x^5) - 5(V_0x^3+V_1x^4+V_2x^5) - 7(V_0x^4+V_1x^5) - 11V_0x^5 \\
&\quad + \sum_{n=6}^{\infty}(V_n-2V_{n-1}-3V_{n-2}-5V_{n-3}-7V_{n-4}-11V_{n-5}-13V_{n-6})x^n \\
&= V_0 + (V_1-2V_0)x + (V_2-2V_1-3V_0)x^2 + (V_3-2V_2-3V_1-5V_0)x^3 \\
&\quad + (V_4-2V_3-3V_2-5V_1-7V_0)x^4 + (V_5-2V_4-3V_3-5V_2-7V_1-11V_0)x^5 \\
&= V_0 + \sum_{i=1}^{6-1}\Big( \quad \Big) x^i
\end{aligned}
$$

$$V_i \qquad \sum_{i=1} \; r_j V_{i \quad j} \qquad : \qquad j=1$$

Rearranging above equation, we obtain (2.1).

The previous lemma gives the following results as particular examples.

Corollary 3. Generated functions of 6-primes, Lucas 6-primes and modi…ed 6-primes numbers are

$$\sum_{n=0}^{\infty} H_n x^n = \frac{x^4}{1 \quad 2x \quad 3x^2 \quad 5x^3 \quad 7x^4 \quad 11x^5 \quad 13x^6};$$

and and

$$\sum_{n=0}^{\infty} H_n x^n = \frac{6 \quad 10x \quad 12x^2 \quad 15x^3 \quad 14x^4 \quad 11x^5}{1 \quad 2x \quad 3x^2 \quad 5x^3 \quad 7x^4 \quad 11x^5 \quad 13x^6};$$

respective

ly.

$$\sum_{n=0}^{\infty} E_n x^n = \frac{x^4 \quad x^5}{1 \quad 2x \quad 3x^2 \quad 5x^3 \quad 7x^4 \quad 11x^5 \quad 13x^6};$$

## 3. Obtaining Binet Formula From Generating Function

We next …nd Binet formula of generalized 6-primes numbers $\{V_n\}$ by the use of generating function for $V_n$:

**Theorem 4.** (Binet formula of generalized 6-primes numbers)

$$V_n = \sum_{k=1}^{6} \frac{d_k \alpha_k^n}{\prod_{j=1, k \neq j}^{6} (\alpha_k - \alpha_j)} \qquad (3.1)$$

where

$$d_1 = V_0 \alpha_1^{6-1} + \sum_{i=1}^{6-1} \left[ V_i - \sum_{j=1}^{i} r_j V_{i-j} \right] \alpha_1^{6-i-1},$$

$$d_l = V_0 \alpha_l^{6-1} + \sum_{i=1}^{6-1} \left[ V_i - \sum_{j=1}^{i} r_j V_{i-j} \right] \alpha_l^{6-i-1}, \quad 1 \leq l \leq m = 6,$$

$$r_1 = 2, r_2 = 3, r_3 = 5, r_4 = 7, r_5 = 11, r_6 = 13.$$

*Proof.* Let

$$h(x) = 1 - 2x - 3x^2 - 5x^3 - 7x^4 - 11x^5 - 13x^6.$$

Then for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and $\alpha_6$ we write

$$h(x) = (1-\alpha_1 x)(1-\alpha_2 x)(1-\alpha_3 x)(1-\alpha_4 x)(1-\alpha_5 x)(1-\alpha_6 x)$$

i.e.,

$$1 - 2x - 3x^2 - 5x^3 - 7x^4 - 11x^5 - 13x^6 = (1-\alpha_1 x)(1-\alpha_2 x)(1-\alpha_3 x)(1-\alpha_4 x)(1-\alpha_5 x)(1-\alpha_6 x) \qquad (3.2)$$

Hence $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \frac{1}{\alpha_4}, \frac{1}{\alpha_5}$ and $\frac{1}{\alpha_6}$ are the roots of $h(x)$. This gives $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and $\alpha_6$ as the roots of

$$h\left(\frac{1}{x}\right) = 1 - \frac{2}{x} - \frac{3}{x^2} - \frac{5}{x^3} - \frac{7}{x^4} - \frac{11}{x^5} - \frac{13}{x^6} = 0.$$

This implies $x^6 - 2x^5 - 3x^4 - 5x^3 - 7x^2 - 11x - 13 = 0$. Now, by (2.1) and (3.2), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + \sum_{i=1}^{6-1} x^i \left( V_i - \sum_{j=1}^{i} r_j V_{i-j} \right)}{(1-\alpha_1 x)(1-\alpha_2 x)(1-\alpha_3 x)(1-\alpha_4 x)(1-\alpha_5 x)(1-\alpha_6 x)}.$$

Then we write

$$\frac{V_0 + \sum_{i=1}^{6-1} x^i \left( V_i - \sum_{j=1}^{i} r_j V_{i-j} \right)}{(1-\alpha_1 x)(1-\alpha_2 x)(1-\alpha_3 x)(1-\alpha_4 x)(1-\alpha_5 x)(1-\alpha_6 x)} = \frac{A_1}{(1-\alpha_1 x)} + \frac{A_2}{(1-\alpha_2 x)} + \frac{A_3}{(1-\alpha_3 x)} + \frac{A_4}{(1-\alpha_4 x)} + \frac{A_5}{(1-\alpha_5 x)} + \frac{A_6}{(1-\alpha_6 x)}. \qquad (3.3)$$

So

$$V_0 + \sum_{i=1} V_i x^i \sum_{j=1} r_j V_{i\ j}$$

$$= A_1 (1\ \ _2 x)(1\ \ _3 x)(1\ \ _4 x)(1\ \ _5 x)(1\ \ _6 x) + A_2 (1\ \ _1 x)(1\ \ _3 x)(1\ \ _4 x)(1\ \ _5 x)(1\ \ _6 x)$$

$$+ A_3 (1\ \ _1 x)(1\ \ _2 x)(1\ \ _4 x)(1\ \ _5 x)(1\ \ _6 x) + A_4 (1\ \ _1 x)(1\ \ _2 x)(1\ \ _3 x)(1\ \ _5 x)(1\ \ _6 x)$$

$$+ A_5 (1\ \ _1 x)(1\ \ _2 x)(1\ \ _3 x)(1\ \ _4 x)(1\ \ _6 x) + A_6 (1\ \ _1 x)(1\ \ _2 x)(1\ \ _3 x)(1\ \ _4 x)(1\ \ _5 x):$$

If we consider $x = \frac{1}{_1}$; we get

$$V_0 + \sum_{i=1}^{6} \frac{1}{_1^{\,i}} \left[ V_i \sum_{j=1}^{i} r_j V_{i\ j} \right] = A_1 (1\ \frac{_2}{_1})(1\ \frac{_3}{_1})(1\ \frac{}{_1})(1\ \frac{}{_1})(1\ \frac{_6}{_1}):$$

4

This gives

$$A_1 = \frac{\frac{1}{\alpha_1}\left(V_0\alpha_1^{6} + \sum_{i=1}^{5}\alpha_1^{6-i}\left(V_i - \sum_{j=1}^{i} r_j V_{i-j}\right)\right)}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)(\alpha_1-\alpha_5)(\alpha_1-\alpha_6)}$$

$$= \frac{V_0\alpha_1^{6-1} + \sum_{i=1}^{6-1}\alpha_1^{6-1-i}\left[V_i - \sum_{j=1}^{i} r_j V_{i-j}\right]}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)(\alpha_1-\alpha_5)(\alpha_1-\alpha_6)}$$

Similarly, we obtain

$$A_2 = \frac{V_0\alpha_2^{6-1} + \sum_{i=1}^{6-1}\alpha_2^{6-1-i}\left[V_i - \sum_{j=1}^{i} r_j V_{i-j}\right]}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)(\alpha_2-\alpha_5)(\alpha_2-\alpha_6)}$$

$$A_3 = \frac{V_0\alpha_3^{6-1} + \sum_{i=1}^{6-1}\alpha_3^{6-1-i}\left[V_i - \sum_{j=1}^{i} r_j V_{i-j}\right]}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)(\alpha_3-\alpha_4)(\alpha_3-\alpha_5)(\alpha_3-\alpha_6)}$$

$$A_4 = \frac{V_0\alpha_4^{6-1} + \sum_{i=1}^{6-1}\alpha_4^{6-1-i}\left[V_i - \sum_{j=1}^{i} r_j V_{i-j}\right]}{(\alpha_4-\alpha_1)(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)(\alpha_4-\alpha_5)(\alpha_4-\alpha_6)}$$

$$A_5 = \frac{V_0\alpha_5^{6-1} + \sum_{i=1}^{6-1}\alpha_5^{6-1-i}\left[V_i - \sum_{j=1}^{i} r_j V_{i-j}\right]}{(\alpha_5-\alpha_1)(\alpha_5-\alpha_2)(\alpha_5-\alpha_3)(\alpha_5-\alpha_4)(\alpha_5-\alpha_6)}$$

$$A_6 = \frac{V_0\alpha_6^{6-1} + \sum_{i=1}^{6-1}\alpha_6^{6-1-i}\left[V_i - \sum_{j=1}^{i} r_j V_{i-j}\right]}{(\alpha_6-\alpha_1)(\alpha_6-\alpha_2)(\alpha_6-\alpha_3)(\alpha_6-\alpha_4)(\alpha_6-\alpha_5)}$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} V_n x^n = A_1(1-\alpha_1 x)^{-1} + A_2(1-\alpha_2 x)^{-1} + A_3(1-\alpha_3 x)^{-1} + A_4(1-\alpha_4 x)^{-1} + A_5(1-\alpha_5 x)^{-1} + A_6(1-\alpha_6 x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} V_n x^n = A_1\sum_{n=0}^{\infty}\alpha_1^n x^n + A_2\sum_{n=0}^{\infty}\alpha_2^n x^n + A_3\sum_{n=0}^{\infty}\alpha_3^n x^n + A_4\sum_{n=0}^{\infty}\alpha_4^n x^n + A_5\sum_{n=0}^{\infty}\alpha_5^n x^n + A_6\sum_{n=0}^{\infty}\alpha_6^n x^n$$

$$= \sum_{n=0}^{\infty}\left(A_1\alpha_1^n + A_2\alpha_2^n + A_3\alpha_3^n + A_4\alpha_4^n + A_5\alpha_5^n + A_6\alpha_6^n\right)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1\alpha_1^n + A_2\alpha_2^n + A_3\alpha_3^n + A_4\alpha_4^n + A_5\alpha_5^n + A_6\alpha_6^n$$

and then we get (3.1).

Next, using Theorem 4, we present the Binet formulas of 6-primes, Lucas 6-primes and modified 6-primes sequences.

**Corollary 5.** Binet formulas of 6-primes, Lucas 6-primes and modified 6-primes sequences are

$$G_n = \sum_{k=1}^{6}\frac{\alpha_k^{n+1}}{\prod_{j=1, k\neq j}^{6}(\alpha_k-\alpha_j)},$$

$$H_n = \sum_{k=1}^{6}\alpha_k^n = \alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n + \alpha_5^n + \alpha_6^n$$

k = 1 + 2 + 3 + 4 + 5 + 6 ;

En =

6 ( k 1) k n+1 ;

k=1

6 Q ( k j)

j=1 k=j

respectively.

## 4. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized 6-primes sequence $\{V_n\}_{n\geq 0}$.

**Theorem 6** (Simson Formula of Generalized 6-primes Numbers). For all integers $n$, we have

$$\begin{vmatrix} V_{n+5} & V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} & V_{n-5} \end{vmatrix} = (-1)^n 13^n \begin{vmatrix} V_5 & V_4 & V_3 & V_2 & V_1 & V_0 \\ V_4 & V_3 & V_2 & V_1 & V_0 & V_{-1} \\ V_3 & V_2 & V_1 & V_0 & V_{-1} & V_{-2} \\ V_2 & V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} \\ V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} \\ V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} & V_{-5} \end{vmatrix}.$$

*Proof.* It is given in Soykan [7].

The previous theorem gives the following results as particular examples.

**Corollary 7.** For all integers $n$, Simson formula of 6-primes, Lucas 6-primes and modified 6-primes numbers are given as

$$\begin{vmatrix} G_{n+5} & G_{n+4} & G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+4} & G_{n+3} & G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+3} & G_{n+2} & G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} & G_{n-2} & G_{n-3} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} & G_{n-3} & G_{n-4} \\ \cdots \end{vmatrix} = (-1)^{n+1} \cdots$$

$13^n$ 4 (4.1)

$G_n$

and

$H_{n+5}$ $H_{n+4}$ $H_{n+3}$
$H_{n+2}$ $H_{n+1}$ $H_n$

$H_{n+4}$ $H_{n+3}$ $H_{n+2}$
$H_{n+1}$ $H_n$ $H_{n-1}$ = 99191747 $2^5$ 41 ( $n+1$ $13^n$ 5 1) (4.2)

$H_{n+3}$ $H_{n+2}$ $H_{n+1}$ $H_n$
$H_{n-1}$ $H_{n-2}$

$H_{n+2}$ $H_{n+1}$ $H_n$ $H_n$
1 $H_{n-2}$ $H_{n-3}$

$H_{n+1}$ $H_n$ $H_{n-1}$ $H_{n-2}$
$H_{n-3}$ $H_{n-4}$

$H_n$ $H_{n-1}$ $H_{n-2}$ $H_{n-3}$ $H_{n-4}$ $H_{n-5}$

and

$$\begin{matrix} E_{n+5} \\ E_{n+4} \\ E_{n+3} \\ E_{n+2} \\ E_{n+1} \\ E_n \end{matrix} \quad \begin{matrix} E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n\ 1} \\ E_{n+2} & E_{n+1} & E_n & E_{n\ 1} & E_{n\ 2} \\ E_{n+1} & E_n & E_{n\ 1} & E_{n\ 2} & E_{n\ 3} \\ E_n & E_{n\ 1} & E_{n\ 2} & E_{n\ 3} & E_{n\ 4} \\ E_{n\ 1} & E_{n\ 2} & E_{n\ 3} & E_{n\ 4} & E_{n\ 5} \end{matrix} \quad 5 = 2^3 \; 1) \quad ( \; n+1 \; 13^n \; 5 \tag{4.3}$$

respectively.

## 5. Some Identities

In this section, we obtain some identities of 6-primes, Lucas 6-primes and modified 6-primes numbers. First, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 8. The following equalities are true:

$$169H_n = 61G_{n+6} \quad 21G_{n+5} + 1483G_{n+4} \quad 956G_{n+3} \quad 886G_{n+2} \quad 863G_{n+1}; \tag{5.1}$$

$$13H_n = 11G_{n+5} + 100G_{n+4} \quad 97G_{n+3} \quad 101G_{n+2} \quad 118G_{n+1} \quad 61G_n;$$

$$H_n = 6G_{n+4} \quad 10G_{n+3} \quad 12G_{n+2} \quad 15G_{n+1} \quad 14G_n \quad 11G_{n\ 1};$$

$$H_n = 2G_{n+3} + 6G_{n+2} + 15G_{n+1} + 28G_n + 55G_{n\ 1} + 78G_{n\ 2};$$

$$H_n =$$

$$H_n = 10G_{n+2} + 21G_{n+1} + 38G_n + 69G_{n\ 1} + 100G_{n\ 2} + 26G_{n\ 3};$$

and

$$65069786032G_n = 17165493H_{n+6} \quad 48224301H_{n+5} + 84682036H_{n+4} \quad 663056677H_{n+3}$$
$$+802372816H_{n+2} \quad 86032399H_{n+1};$$

$$65069786032G_n = 13893315H_{n+5} + 136178515H_{n+4} \quad 577229212H_{n+3} + 922531267H_{n+2}$$
$$+102788024H_{n+1} + 223151409H_n;$$

$$65069786032G_n = 108391885H_{n+4} \quad 618909157H_{n+3} + 853064692H_{n+2} + 5534819H_{n+1}$$

$$+70324944H_{n} \quad 180613095H_{n} \quad 1;$$

$$65069786032G_{n} = 402125387H_{n+3} + 1178240347H_{n+2} + 547494244H_{n+1} + 829068139H_{n}$$

$$+1011697640H_{n} \quad 1 + 1409094505H_{n} \quad 2;$$

$$65069786032G_{n} = 373989573H_{n+2} \quad 658881917H_{n+1} \quad 1181558796H_{n} \quad 1803180069H_{n} \quad 1$$

$$3014284752H_{n} \quad 2 \quad 5227630031H_{n} \quad 3;$$

$$65069786032G_{n} = 89097229H_{n+1} \quad 59590077H_{n} + 66767796H_{n} \quad 1 \quad 396357741H_{n} \quad 2$$

$$1113744728H_{n} \quad 3 + 4861864449H_{n} \quad 4:$$

Proof. Note that all the identities hold for all integers n: We prove (5.1). To show (5.1), writing

$$H_n = a \quad G_{n+6} + b \quad G_{n+5} + c \quad G_{n+4} + d \quad G_{n+3} + e \quad G_{n+2} + f \quad G_{n+1}$$

and solving the system of equations

$$\begin{aligned}
H_0 &= a \quad G_6 + b \quad G_5 + c \quad G_4 + d \quad G_3 + e \quad G_2 + f \quad G_1\\
H_1 &= a \quad G_7 + b \quad G_6 + c \quad G_5 + d \quad G_4 + e \quad G_3 + f \quad G_2\\
H_2 &= a \quad G_8 + b \quad G_7 + c \quad G_6 + d \quad G_5 + e \quad G_4 + f \quad G_3\\
H_3 &= a \quad G_9 + b \quad G_8 + c \quad G_7 + d \quad G_6 + e \quad G_5 + f \quad G_4\\
H_4 &= a \quad G_{10} + b \quad G_9 + c \quad G_8 + d \quad G_7 + e \quad G_6 + f \quad G_5\\
H_5 &= a \quad G_{11} + b \quad G_{10} + c \quad G_9 + d \quad G_8 + e \quad G_7 + f \quad G_6
\end{aligned}$$

we find that $a = \frac{61}{169}$; $b = \frac{21}{169}$; $c = \frac{1483}{169}$; $d = \frac{956}{169}$; $e = \frac{886}{169}$; $f = \frac{863}{169}$: The other equalities can be proved similarly.

Secondly, we present a few basic relations between $\{G_n\}$ and $\{E_n\}$.

Lemma 9. The following equalities are true:

$$\begin{aligned}
169E_n &= 24G_{n+6} \quad 61G_{n+5} \quad 46G_{n+4} \quad 81G_{n+3} \quad 103G_{n+2} \quad 173G_{n+1};\\
13E_n &= G_{n+5} + 2G_{n+4} + 3G_{n+3} + 5G_{n+2} + 7G_{n+1} + 24G_n;\\
E_n &= G_n \quad G_{n-1};
\end{aligned}$$

and

$$\begin{aligned}
40G_n &= E_{n+6} \quad E_{n+5} \quad 4E_{n+4} \quad 9E_{n+3} \quad 16E_{n+2} \quad 27E_{n+1};\\
40G_n &= E_{n+5} \quad E_{n+4} \quad 4E_{n+3} \quad 9E_{n+2} \quad 16E_{n+1} + 13E_n;\\
40G_n &= E_{n+4} \quad E_{n+3} \quad 4E_{n+2} \quad 9E_{n+1} + 24E_n + 13E_{n-1};\\
40G_n &= E_{n+3} \quad E_{n+2} \quad 4E_{n+1} + 31E_n + 24E_{n-1} + 13E_{n-2};\\
40G_n &= E_{n+2} \quad E_{n+1} + 36E_n + 31E_{n-1} + 24E_{n-2} + 13E_{n-3};\\
40G_n &= E_{n+1} + 39E_n + 36E_{n-1} + 31E_{n-2} + 24E_{n-3} + 13E_{n-4};
\end{aligned}$$

Note that all the identities in the above Lemma can be proved by induction as well. Thirdly, we give a few basic relations between $\{H_n\}$

and fEn g.

Lemma 10. The following equalities are true:

$$65H_n = 36E_{n+6} \quad 19E_{n+5} + 589E_{n+4} + 284E_{n+3} + 31E_{n+2} \quad 163E_{n+1};$$

$$5H_n = 7E_{n+5} + 37E_{n+4} + 8E_{n+3} \quad 17E_{n+2} \quad 43E_{n+1} \quad 36E_n;$$

$$5H_n = 23E_{n+4} \quad 13E_{n+3} \quad 52E_{n+2} \quad 92E_{n+1} \quad 113E_n \quad 91E_{n\ 1};$$

$$5H_n = 33E_{n+3} + 17E_{n+2} + 23E_{n+1} + 48E_n + 162E_{n\ 1} + 299E_{n\ 2};$$

$$5H_n = 83E_{n+2} + 122E_{n+1} + 213E_n + 393E_{n\ 1} + 662E_{n\ 2} + 429E_{n\ 3};$$

and

$$105738402302E_n = 38647976H_{n+6} \quad 127766515H_{n+5} + 183710648H_{n+4} \quad 1268845658H_{n+3} +2306044577H_{n+2} \quad 1561953023H_{n+1};$$

$$8133723254E_n = 3882351H_{n+5} + 23050352H_{n+4} \quad 82738906H_{n+3} + 198198493H_{n+2} \quad 87448099H_{n+1} + 38647976H_n;$$

$$8133723254E_n = 15285650H_{n+4} \quad 94385959H_{n+3} + 178786738H_{n+2} \quad 114624556H_{n+1} \quad 4057885H_n \quad 50470563H_{n\ 1};$$

$$8133723254E_n = 63814659H_{n+3} + 224643688H_{n+2} \quad 38196306H_{n+1} + 102941665H_n +117671587H_{n\ 1} + 198713450H_{n\ 2};$$

$$8133723254E_n = 97014370H_{n+2} \quad 229640283H_{n+1} \quad 216131630H_n \quad 329031026H_{n\ 1} \quad 503247799H_{n\ 2} \quad 829590567H_{n\ 3};$$

$$8133723254E_n = 35611543H_{n+1} + 74911480H_n + 156040824H_{n\ 1} + 175852791H_{n\ 2} +237567503H_{n\ 3} + 1261186810H_{n\ 4}:$$

We now present a few special identities for the modi…ed 6-primes sequence $\{E_n\}$.

Theorem 11. (Catalan's identity) For all integers $n$ and $m$; the following identity holds

$$E_{n+m}E_{n\ m} \quad E_n^2 = (G_{n+m} \quad G_{n+m\ 1})(G_{n\ m} \quad G_{n\ m\ 1}) \quad (G_n \quad G_{n\ 1})^2$$

$$= (G_n(G_m \quad G_{m+1}) + G_{n\ 1}(\ G_m + G_{m\ 2}) + G_{n\ 2}(\ G_m + G_{m\ 1}))$$

$$(G_n(G_{\ m} \quad G_{1\ m}) + G_{n\ 1}(\ G_{\ m} + G_{\ m\ 2}) + G_{n\ 2}(\ G_{\ m} + G_{\ m\ 1}))$$

$$(G_n \quad G_n^2 \quad 1):$$

Proof. We use the identity

$$E_n = G_n \quad G_n \quad 1:$$

Note that for m = 1 in Catalan's identity, we get the Cassini identity for the modi…ed 6-primes sequnce

**Corollary 12.** (Cassini's identity) For all integers numbers $n$ and $m$; the following identity holds

$$E_{n+1}E_{n-1}-E_n^2=(G_{n+1}-G_n)(G_{n-1}-G_{n-2})-(G_n-G_{n-1})^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using $E_n=G_n-G_{n-1}$. The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified 6-primes sequence $\{E_n\}$.

**Theorem 13.** Let $n$ and $m$ be any integers. Then the following identities are true:

(a): (d'Ocagne's identity)

$$E_{m+1}E_n-E_mE_{n+1}=(G_{m+1}-G_m)(G_n-G_{n-1})-(G_m-G_{m-1})(G_{n+1}-G_n).$$

(b): (Gelin-Cesàro's identity)

$$E_{n+2}E_{n+1}E_{n-1}E_{n-2}-E_n^4=(G_{n+2}-G_{n+1})(G_{n+1}-G_n)(G_{n-1}-G_{n-2})(G_{n-2}-G_{n-3})-(G_n-G_{n-1})^4.$$

(c): (Melham's identity)

$$E_{n+1}E_{n+2}E_{n+6}-E_{n+3}^3=(G_{n+1}-G_n)(G_{n+2}-G_{n+1})(G_{n+6}-G_{n+5})-(G_{n+3}-G_{n+2})^3.$$

*Proof.* Use the identity $E_n=G_n-G_{n-1}$.

## 6. Linear Sums

The following Theorem presents some linear summing formulas of generalized Hexanacci numbers with positive subscripts.

**Theorem 14.** For $n\geq 0$ we have the following formulas:

(a): (Sum of the generalized Hexanacci numbers) If $r_1+r_2+r_3+r_4+r_5+r_6-1\neq 0$ then

$$\sum_{k=0}^{n}W_k=\frac{\Theta_1}{r_1+r_2+r_3+r_4+r_5+r_6-1}$$

where

$$\Theta_1=W_{n+6}+(1-r_1)W_{n+5}+(1-r_1-r_2)W_{n+4}+(1-r_1-r_2-r_3)W_{n+3}$$
$$+(1-r_1-r_2-r_3-r_4)W_{n+2}+(1-r_1-r_2-r_3-r_4-r_5$$

$)W_{n+1}$

$W_5 + (r_1 \quad 1)W_4 + (r_1 + r_2 \quad 1)W_3 + (r_1 + r_2 + r_3 \quad 1)W_2$

$+(r_1 + r_2 + r_3 + r_4 \quad 1)W_1 + (r_1 + r_2 + r_3 + r_4 + r_5 \quad 1)W_0 :$

(b): If $(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 \quad 1)(r_1 \quad r_2 + r_3 \quad r_4 + r_5 \quad r_6 + 1) = 0$ then

$$\sum_{k=0}^{n} W_{2k} = \frac{2}{(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 \quad 1)(r_1 \quad r_2 + r_3 \quad r_4 + r_5 \quad r_6 + 1)}$$

where

$2 = (r_2 + r_4 + r_6 \quad 1) W_{2n+2} + (r_3 + r_5 + r_1 (r_2 + r_4 + r_6)) W_{2n+1}$

$+(r_4 + r_6 + r_1 (r_3 + r_5) \quad r_2 (r_4 + r_6^2 \quad (r_4 + r_6)^2) W_{2n}$
$) + (r_3 + r_5)$

2 2 $+(r_5 \quad r_2 r_5 + (r_1 + r_3) r_4 + (r_1 + r_3) r_6) W_{2n \quad 1} + (r_6 + (r_1 + r_3) r_5 \quad (r_2 + r_4$
$) r_6 + r_5 \quad r_6) W_{2n \quad 2}$

$+r_6 (r_1 + r_3 + r_5) W_{2n \quad 3} \quad (r_1 + r_3 + r_5) W_5 + (r_2 + r_4 + r_6 + (r_1 + r_3 + r_5) r_1$
$1) W_4$

$+((r_3 + r_5) r_2 \quad (r_4 + r_6) r_1 \quad r_3 \quad r_5) W_3$

$+(r_4 + r_6 + (r_1 + r_3) r_5 \quad (r_4 + r_6^{\ 2} \quad (r_2^{\ 2} 1) \ ) W_2 + ( \quad r_5 + (r_2 + r_4) r_5 \quad (r_1 +$
$) r_2 + (r_1 + r_3) \quad r_3) r_6) W_1$

$+(2r_2 + 2r_4 + 2r_1 r_5 + 2r_3 r_5 \quad r_2 r_6 + r_6^{\ 2} \quad (r_2 +^{2} + r_5^2 \quad 1) W :$
$r_4 r_6 + (r_1 + r_3) \quad r_4)$

(c):

$$\sum_{k=0}^{n} W_{2k+1} = \frac{3}{(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 \quad 1)(r_1 \quad r_2 + r_3 \quad r_4 + r_5 \quad r_6 + 1)}$$

where

$3 = (r_1 + r_3 + r_5) W_{2n+2} + ((r_2 + r_4 + r_6)^2 + (r_3 + r_5)^2 + r_1 (r_3 + r_5)) W_{2n+1}$
$(r_2 + r_4 + r_6)$

$+ ((1 \quad r_2)(r_3 + r_5) + r_1 (r_4 + r_6)) W_{2n} + ((r_1 + r_3) r_5 + r_5^2 \quad (r_4 + r_6) r_2^2 + (r_4 + r_6$
$) \quad (r_4 + r_6) \ ) W_{2n \quad 1}$

$+ ((1 \quad (r_2 + r_4)) r_5 + (r_1 + r_3) r_6) W_{2n \quad 2} \quad r_6 (r_2 + r_4 + r_6 \quad 1) W_{2n \quad 3} + (r_2 +$
$r_4 + r_6 \quad 1) W_5$

2 2 $((r_3 + r_5) + (r_2 + r_4 + r_6) r_1) W_4 + (2r_2 + r_4 + r_6 + r_1 r_3 + r_1 r_5 \quad r_2 r_4 \quad r_2 r_6$
$+ r_1 \quad r_2 \quad 1) W_3$

$((1 \quad r_2) r_5 + (r_1 + r_3)(r_4 + r_6)) W_2$

$+(2r_2 + 2r_4 + r_6 + r_1 r_5 + r_3 r_5 \quad r_2^2 r_6^2 \quad r_4 r_6 + r_1^2 + r_3^2 + 2r_1 r_3 \quad r_2 \quad r_4$
$2r_2 r_4 \quad 1) W_1 \quad r_6 (r_1 + r_3 + r_5) W_0 :$

Proof. The proof is given in Soykan [8].

The following proposition presents some formulas of generalized 6-primes numbers with positive subscripts.

**Proposition 15.** If $r_1 = 2; r_2 = 3; r_3 = 5; r_4 = 7; r_5 = 11; r_6 = 13$ then for $n \quad 0$ we have the following formulas:

(a): $\sum_{k=0}^{n} V_k = \frac{1}{40} (V_{n+6} \quad V_{n+5} \quad 4V_{n+4} \quad 9V_{n+3} \quad 16V_{n+2} \quad 27V_{n+1} \quad V_5 + V_4 +$
$4V_3 + 9V_2 + 16V_1 + 27V_0):$

(b): $\sum_{k=0}^{n} V_{2k} = \frac{1}{80} (11V_{2n+2} \quad 31V_{2n+1} + 44V_{2n} + 117V_{2n \quad 2} + 9V_{2n \quad 3} + 29V_4 + 4V_3 \quad 41V_2$
$76V_{2n} \quad 59V_{2n \quad 5}$
$4V_1 \quad 63V_0):$

(c): $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{80}( 9V_{2n+2} + 109V_{2n+1}\ 4V_{2n} + 121V_{2n\ 1} + 4V_{2n\ 2} + 143V_{2n\ 3}\ 11V_{2n\ 4}\ 31V_{2n\ 5} + \ 4 + 4V_3 + 59V_2 + 36V_1 + 117V_0 )$:

Proof. Take $r_1 = 2; r_2 = 3; r_3 = 5; r_4 = 7; r_5 = 11$ in Theorem 14.

As special cases of above proposition, we have the following three corollaries. First one presents some summing formulas of 6-primes numbers (take $V_n = G_n$ with $G_0 = 0; G_1 = 0; G_2 = 0; G_3 = 0; G_4 = 1; G_5 = 2$):

Corollary 16. For $n \geq 0$ we have the following formulas:

(a): $\sum_{k=0}^{n} G_k = \frac{1}{40}(G_{n+6}\ G_{n+5}\ 4G_{n+4}\ 9G_{n+3}\ 16G_{n+2}\ 27G_{n+1}\ 1)$:

(b): $\sum_{k=0}^{n} G_{2k} = \frac{1}{80}(11G_{2n+2}\ 31G_{2n+1} + 76G_{2n}\ 59G_{2n\ 1} + 44G_{2n\ 2} + 117G_{2n\ 3}\ 11)$:

(c): $\sum_{k=0}^{n} G_{2k+1} = \frac{1}{80}( 9G_{2n+2} + 109G_{2n+1}\ 4G_{2n} + 121G_{2n\ 1} + 4G_{2n\ 2} + 143G_{2n\ 3} + 9)$:

Second one presents some summing formulas of Lucas 6-primes numbers (take $G_n = H_n$ with $H_0 = 6; H_1 = 2; H_2 = 10; H_3 = 41; H_4 = 150; H_5 = 542$):

**Corollary 17.** For $n \geq 0$ we have the following formulas:

(a): $\sum_{k=0}^{n} H_k = \frac{1}{40}(H_{n+6} - H_{n+5} - 4H_{n+4} - 9H_{n+3} - 16H_{n+2} - 27H_{n+1} + 56)$:

(b): $\sum_{k=0}^{n} H_{2k} = \frac{1}{80}(11H_{2n+2} - 31H_{2n+1} + 76H_{2n} - 59H_{2n-1} + 44H_{2n-2} - 117H_{2n-3} - 104)$:

(c): $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{80}(-9H_{2n+2} + 109H_{2n+1} - 4H_{2n} + 121H_{2n-1} + 4H_{2n-2} + 143H_{2n-3} + 216)$:

Third one presents some summing formulas of modified 6-primes numbers (take $H_n = E_n$ with $E_0 = 0; E_1 = 0; E_2 = 0; E_3 = 0; E_4 = 1; E_5 = 1$).

**Corollary 18.** For $n \geq 0$ we have the following formulas:

(a): $\sum_{k=0}^{n} E_k = \frac{1}{40}(E_{n+6} - E_{n+5} - 4E_{n+4} - 9E_{n+3} - 16E_{n+2} - 27E_{n+1})$:

(b): $\sum_{k=0}^{n} E_{2k} = \frac{1}{80}(11E_{2n+2} - 31E_{2n+1} + 76E_{2n} - 59E_{2n-1} + 44E_{2n-2} - 117E_{2n-3} - 20)$:

(c): $\sum_{k=0}^{n} E_{2k+1} = \frac{1}{80}(-9E_{2n+2} + 109E_{2n+1} - 4E_{2n} + 121E_{2n-1} + 4E_{2n-2} + 143E_{2n-3} + 20)$:

The following Theorem presents some linear summing formulas of generalized Hexanacci numbers with negative subscripts.

**Theorem 19.** For $n \geq 1$ we have the following formulas:

(a): (Sum of the generalized Hexanacci numbers with negative indices) If $r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - 1 = 0$;

then

$$\sum_{k=1}^{n} W_{-k} = \frac{4}{r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - 1}$$

where

$$4 = W_{n+5} + (r_1 \quad 1)W_{n+4} + (r_1 + r_2 \quad 1)W_{n+3} + (r_1 + r_2 + r_3 \quad 1)W_{n+2}$$

$$+(r_1 + r_2 + r_3 + r_4 \quad 1)W_{n+1} + (r_1 + r_2 + r_3 + r_4 + r_5 \quad 1)W_n$$

$$+W_5 + (1 \quad r_1)W_4 + (1 \quad r_1 \quad r_2)W_3 + (1 \quad r_1 \quad r_2 \quad r_3)W_2$$

$$+(1 \quad r_1 \quad r_2 \quad r_3 \quad r_4)W_1 + (1 \quad r_1 \quad r_2 \quad r_3 \quad r_4 \quad r_5)W_0 :$$

(b): If $(r_1 \quad r_2 + r_3 \quad r_4 + r_5 \quad r_6 + 1)(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 \quad 1) = 0$ then

$$\sum_{k=1}^{n} W_{2k} = \frac{5}{(r_1 \quad r_2 + r_3 \quad r_4 + r_5 \quad r_6 + 1)(r_1 \quad + r_2 + r_3 + r_4 + r_5 + r_6 \quad 1)}$$

where

$5 = (r_2 + r_4 + r_6 \quad 1)W \quad 2n{+}4 \quad (r_3 + r_5 + (r_2 + r_4 + r_6)r_1)W \quad 2n{+}3$

$+(r_4 + r_6 \quad (r_4 + r_6)r_2 + (r_3 + r_5 + r_1)r_1 \quad (r_2 \quad 1)^2)W \quad 2n{+}2$

$((r_1 + r_3)r_4 + (r_1 + r_3)r_6 + (1 \quad r_2)r_5)W \quad 2n{+}1$

$+(r_6 + 2(r_2 + r_4) + (r_1 + r_3)r_5 \quad (r_2 + r_4 \quad ^2 \quad (r_2 + \quad ^2 \quad 1)W \quad 2n$
$)r_6 + (r_1 + r_3) \quad r_4)$

$r_6(r_1 + r_3 + r_5)W \quad 2n \quad 1 + (r_1 + r_3 + r_5)W_5 \quad (r_2 + r_4 + r_6 + (r_3 + r_5 + r_1$
$)r_1 \quad 1)W_4$

$+((r_4 + r_6)r_1 + (1 \quad r_2)(r_3 + r_5))W_3 + ( \quad r_4 \quad r_6 \quad (r_1 + r_3)r_5 \quad ^2 + (r_2 \quad 1)^2$
$+ (r_4 + r_6)r_2 \quad (r_1 + r_3) \quad )W_2$

$+(r_5 + (r_1 + r_3)r_6 \quad (r_2 + r_4)r_5)W_1$

$+( \quad r_6 \quad 2(r_2 + r_4) \quad 2(r_1 + r_3)r_5 + (r_2 + \quad ^2 + (r_2 + \quad r_5 \quad ^2 + 1)W_0 :$
$r_4)r_6 \quad (r_1 + r_3) \quad r_4)^2$

(c):

$\sum_{k=1}^{n} W_{2k+1} = \frac{6}{(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + 1)(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 \quad 1)}$

where

$6 = (r_1 + r_3 + r_5)W \quad 2n{+}4 + (r_2 + r_4 + r_6 + (r_3 + r_5 + r_1)r_1 \quad 1)W \quad 2n{+}3$

$+ ((r_2 \quad 1)(r_3 + r_5) \quad (r_4 + r_6)r_1)W \quad 2n{+}2$

$+(r_4 + r_6 + (r_1 + r_3)r_5 \quad (r_4 + r_6 \quad ^2 \quad (r_2 \quad ^2 \, 1))W \quad 2n{+}1$
$)r_2 + (r_1 + r_3)$

$+( \quad r_5 \quad (r_1 + r_3)r_6 + (r_2 + r_4)r_5)W \quad 2n + r_6(r_2 + r_4 + r_6 \quad 1)W \quad 2n \quad 1$
$(r_2 + r_4 + r_6 \quad 1)W_5$

$+(r_3 + r_5 + (r_2 + r_4 + r_6)r_1)W_4 + ( \quad r_4 \quad r_6 \quad (r_1 + r_3 + r_5)r_1 + (r_4 + r_6$
$)r_2 + (r_2 \quad 1)^2)W_3$

$+((r_4 + r_6)r_1 + (r_4 + r_6)r_3 + (1 \quad r_2)r_5)W_2$

$+( \quad r_6 \quad 2(r_2 + r_4) \quad (r_1 + r_3)r_5 + (r_2 + \quad ^2 + (r_2 + \quad + 1)W_1 + r_6(r_1 + r_3 + r_5$
$r_4)r_6 \quad (r_1 + r_3) \quad r_4)^2 \quad )W_0 :$

Proof. The proof is given in Soykan [8].

The following proposition presents some formulas of generalized 6-primes numbers with negative subscripts.

**Proposition 20.** If $r_1 = 2; r_2 = 3; r_3 = 5; r_4 = 7; r_5 = 11; r_6 = 13$ then for $n \quad 1$ we have the following formulas:

(a): $\sum_{k=1}^{n} V_{k} = \frac{1}{480}( +V_{n+5} +4V_{n+4} +9V_{n+3} +16V_{n+2} +27V_{n+1} +27V_{n} \quad ): V_5 \quad 4V_4 \quad 9V_3 \quad 16V_2 \quad 1V_1 \quad 0$

(b): $\sum_{k=1}^{n} G_{2k} = \frac{1}{}( \quad 11V \quad 2n{+}4 + 31V \quad 2n{+}3$

$+ 4V_{2n+2} + 59V_{2n+1} + 36V_{2n} + 117V_{2n-1} + 9V_5 + 29V_4 + 4V_3 + 41V_2 + 4V_1 + 63V_0)$:

(c): $\sum_{k=1}^{n} G_{2k+1} = \frac{1}{80}(9V_{2n+4} + 29V_{2n+3} + 4V_{2n+2} + 41V_{2n+1} + 4V_{2n} + 143V_{2n-1} + 11V_5 + 31V_4$

$4V_3 \quad 59V_2 \quad 36V_1 \quad 117V_0)$:

Proof. Take $r_1 = 2; r_2 = 3; r_3 = 5; r_4 = 7; r_5 = 11$ in Theorem 19.

From the above proposition, we have the following corollary which gives sum formulas of 6-primes numbers (take

$G_n = G_n$ with $G_0 = 0; G_1 = 0; G_2 = 0; G_3 = 0; G_4 = 1; G_5 = 2)$:

Corollary 21. For $n \ge 1$; 6-primes numbers have the following properties.

(a): $\sum_{k=0}^{n} G_k = \frac{1}{4}(G_{n+5} + 4G_{n+4} + 9G_{n+3} + 16G_{n+2} + 27G_{n+1} \; G_n + 1)$:

(b): $\sum_{k=1}^{n} G_{2k} = \frac{1}{80}(-11G_{2n+4} + 31G_{2n+3} + 4G_{2n+2} + 59G_{2n+1} + 36G_{2n} + 117G_{2n-1} + 11)$:

(c): $\sum_{k=1}^{n} G_{2k+1} = \frac{1}{80}(9G_{2n+4} - 29G_{2n+3} + 4G_{2n+2} - 41G_{2n+1} - 4G_{2n} - 143G_{2n-1} - 9)$:

Taking $G_n = H_n$ with $H_0 = 6; H_1 = 2; H_2 = 10; H_3 = 41; H_4 = 150; H_5 = 542$ in the last proposition, we have the following corollary which presents sum formulas of 6-primes -Lucas numbers.

**Corollary 22.** For $n \geq 1$; 6-primes -Lucas numbers have the following properties.

(a): $\sum_{k=1}^{n} H_k = \frac{1}{4}(-H_{n+5} + H_{n+4} + 4H_{n+3} + 9H_{n+2} + 16H_{n+1} + 27H_n - 56)$:

(b): $\sum_{k=1}^{n} H_{2k} = \frac{1}{80}(-11H_{2n+4} + 31H_{2n+3} + 4H_{2n+2} + 59H_{2n+1} + 36H_{2n} + 117H_{2n-1} + 104)$:

(c): $\sum_{k=1}^{n} H_{2k+1} = \frac{1}{80}(9H_{2n+4} - 29H_{2n+3} + 4H_{2n+2} - 41H_{2n+1} - 4H_{2n} - 143H_{2n-1} - 216)$:

From the above proposition, we have the following corollary which gives sum formulas of modified 6-primes numbers (take $H_n = E_n$ with $E_0 = 0; E_1 = 0; E_2 = 0; E_3 = 0; E_4 = 1; E_5 = 1$).

**Corollary 23.** For $n \geq 1$; modified 6-primes numbers have the following properties.

(a): $\sum_{k=1}^{n} E_k = \frac{1}{4}(-E_{n+5} + E_{n+4} + 4E_{n+3} + 9E_{n+2} + 16E_{n+1} + 27E_n)$:

(b): $\sum_{k=1}^{n} E_{2k} = \frac{1}{80}(-11E_{2n+4} + 31E_{2n+3} + 4E_{2n+2} + 59E_{2n+1} + 36E_{2n} + 117E_{2n-1} + 20)$:

(c): $\sum_{k=1}^{n} E_{2k+1} = \frac{1}{80}(9E_{2n+4} - 29E_{2n+3} + 4E_{2n+2} - 41E_{2n+1} - 4E_{2n} - 143E_{2n-1} - 20)$:

## 7. Matrices Related with Generalized 6-primes Numbers

Matrix formulation of $W_n$ can be given as

$$\begin{pmatrix} W_{n+5} \\ W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^n \begin{pmatrix} W_5 \\ W_4 \\ W_3 \end{pmatrix} \quad (7.1)$$

$$\begin{matrix} 0 & 0 & 0 & 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} W_2 \\ W_1 \\ W_0 \end{matrix}$$

For matrix formulation (7.1), see [2]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \\ W_{n+4} \\ W_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{pmatrix}^{n} \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \end{pmatrix}.$$

We define the square matrix A of order 6 as:

$$A = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = -13$. From (1.4) we have

$$\begin{pmatrix} V_{n+5} \\ V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix}. \quad (7.2)$$

and from (7.1) (or using (7.2) and induction) we have

$$\begin{pmatrix} V_{n+5} \\ V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_5 \\ V_4 \\ V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V_n = G_n$ in (7.2) we have

$$\begin{pmatrix} G_{n+5} \\ G_{n+4} \\ G_{n+3} \\ \vdots \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \vdots \end{pmatrix} \begin{pmatrix} G_{n+4} \\ G_{n+3} \\ G_{n+2} \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} \vdots \\ G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} \vdots & & & & & \vdots \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \vdots \\ G_{n+1} \\ G_n \\ G_{n-1} \end{pmatrix}. \quad (7.3)$$

We also define

$$B_n = \begin{pmatrix}
G_{n+4} & \sum_{k=0}^{5} r_{k+2}G_{n+3-k} & \sum_{k=0}^{4} r_{k+3}G_{n+3-k} & \sum_{k=0}^{3} r_{k+4}G_{n+3-k} & \sum_{k=0}^{2} r_{k+5}G_{n+3-k} & r_6G_{n+3} \\
G_{n+3} & \sum_{k=1}^{6} r_{k+1}G_{n+3-k} & \sum_{k=1}^{5} r_{k+2}G_{n+3-k} & \sum_{k=1}^{4} r_{k+3}G_{n+3-k} & \sum_{k=1}^{3} r_{k+4}G_{n+3-k} & r_6G_{n+2} \\
G_{n+2} & \sum_{k=2}^{7} r_kG_{n+3-k} & \sum_{k=2}^{6} r_{k+1}G_{n+3-k} & \sum_{k=2}^{5} r_{k+2}G_{n+3-k} & \sum_{k=2}^{4} r_{k+3}G_{n+3-k} & r_6G_{n+1} \\
G_{n+1} & \sum_{k=3}^{8} r_{k-1}G_{n+3-k} & \sum_{k=3}^{7} r_kG_{n+3-k} & \sum_{k=3}^{6} r_{k+1}G_{n+3-k} & \sum_{k=3}^{5} r_{k+2}G_{n+3-k} & r_6G_{n} \\
G_{n} & \sum_{k=4}^{9} r_{k-2}G_{n+3-k} & \sum_{k=4}^{8} r_{k-1}G_{n+3-k} & \sum_{k=4}^{7} r_kG_{n+3-k} & \sum_{k=4}^{6} r_{k+1}G_{n+3-k} & r_6G_{n-1} \\
G_{n-1} & \sum_{k=5} r_{k-3}G_{n+3-k} & \sum_{k=5} r_{k-2}G_{n+3-k} & \sum_{k=5} r_{k-1}G_{n+3-k} & \sum_{k=5} r_kG_{n+3-k} & r_6G_{n-2}
\end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+4} & \sum_{k=0}^{4} r_{k+2}V_{n+3-k} & \sum_{k=0}^{3} r_{k+3}V_{n+3-k} & \sum_{k=0}^{2} r_{k+4}V_{n+3-k} & \sum_{k=0}^{1} r_{k+5}V_{n+3-k} & r_6V_{n+3} \\ V_{n+3} & \sum_{k=1}^{5} r_{k+1}V_{n+3-k} & \sum_{k=1}^{4} r_{k+2}V_{n+3-k} & \sum_{k=1}^{3} r_{k+3}V_{n+3-k} & \sum_{k=1}^{2} r_{k+4}V_{n+3-k} & r_6V_{n+2} \\ V_{n+2} & \sum_{k=2}^{6} r_{k}V_{n+3-k} & \sum_{k=2}^{5} r_{k+1}V_{n+3-k} & \sum_{k=2}^{4} r_{k+2}V_{n+3-k} & \sum_{k=2}^{3} r_{k+3}V_{n+3-k} & r_6V_{n+1} \\ V_{n+1} & \sum_{k=3}^{7} r_{k-1}V_{n+3-k} & \sum_{k=3}^{6} r_{k}V_{n+3-k} & \sum_{k=3}^{5} r_{k+1}V_{n+3-k} & \sum_{k=3}^{4} r_{k+2}V_{n+3-k} & r_6V_{n} \\ V_{n} & \sum_{k=4}^{8} r_{k-2}V_{n+3-k} & \sum_{k=4}^{7} r_{k-1}V_{n+3-k} & \sum_{k=4}^{6} r_{k}V_{n+3-k} & \sum_{k=4}^{5} r_{k+1}V_{n+3-k} & r_6V_{n-1} \\ V_{n-1} & \sum_{k=5}^{9} r_{k-3}V_{n+3-k} & \sum_{k=5}^{8} r_{k-2}V_{n+3-k} & \sum_{k=5}^{7} r_{k-1}V_{n+3-k} & \sum_{k=5}^{6} r_{k}V_{n+3-k} & r_6V_{n-2} \end{pmatrix}$$

where

$$r_1 = 2;\ r_2 = 3;\ r_3 = 5;\ r_4 = 7;\ r_5 = 11;\ r_6 = 13.$$

**Theorem 24.** For all integer $m, n \geq 0$, we have

(a): $B_n = A^n$.

(b): $C_1A^n = A^nC_1$.

(c): $C_{n+m} = C_nB_m = B_mC_n$.

*Proof.*

(a): By expanding the vectors on the both sides of (7.3) to 6-colums and multiplying the obtained on the right-hand side by $A$, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b): Using (a) and definition of $C_1$, (b) follows.

(c): We have $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n.$$

Some properties of matrix $A^n$ can be given as

$$A^n = 2A^{n-1} + 3A^{n-2} + 5A^{n-3} + 7A^{n-4} + 11A^{n-5} + 13A^{n-6}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = (-13)^n$$

for all integer m and n:

**Theorem 25.** For $m, n \geq 0$ we have

$$V_{n+m} = V_n G_{m+4} + \sum_{i=1}^{6-1} V_{n-i}\left(\sum_{j=1}^{6-i} r_{j+i} G_{m+4-j}\right) \qquad (7.4)$$

$$= V_n G_{m+4} + V_{n-1}(3G_{m+3} + 5G_{m+2} + 7G_{m+1} + 11G_m + 13G_{m-1})$$
$$+V_{n-2}(5G_{m+3} + 7G_{m+2} + 11G_{m+1} + 13G_m) + V_{n-3}(7G_{m+3} + 11G_{m+2} + 13G_{m+1})$$
$$+V_{n-4}(11G_{m+3} + 13G_{m+2}) + 13V_{n-5}G_{m+3}.$$

*Proof.* From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of $C_{n+m}$ is the product of row $C_n$ and a column $B_m$. From the last equation we say that an element of $C_{n+m}$ is the product of a row $C_n$ and column $B_m$. We just compare the linear combination of the 2nd row and 1st column entries of the matrices $C_{n+m}$ and $C_n B_m$. This completes the proof.

**Remark 26.** By induction, it can be proved that for all integers $m, n \leq 0$, (7.4) holds. So for all integers $m, n$; (7.4) is true.

**Corollary 27.** For all integers $m, n$; we have

$$G_{n+m} = G_n G_{m+4} + \sum_{i=1}^{6-1} G_{n-i}\left(\sum_{j=1}^{6-i} r_{j+i} G_{m+4-j}\right); \qquad (7.5)$$

$$H_{n+m} = H_n G_{m+4} + \sum_{i=1}^{6-1} H_{n-i}\left(\sum_{j=1}^{6-i} r_{j+i} G_{m+4-j}\right); \qquad (7.6)$$

$$E_{n+m} = E_n G_{m+4} + \sum_{i=1}^{6-1} E_{n-i}\left(\sum_{j=1}^{6-i} r_{j+i} G_{m+4-j}\right). \qquad (7.7)$$

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