Review Article

A CONSTRUCTIVE METHOD FOR GENERATING SHORT PRESENTATIONS FOR THE SYMMETRIC GROUPS S_{m+n}, S_{2m} AND S_{mn}

ABSTRACT

A long-standing problem is how to create a short-length presentation for finite groups of degree *n*. This paper aimed at presenting a concrete method for generating presentations for the groups S_{m+n}, S_{2m} and S_{mn} for all $m,n \in Z^+$ with fewer relations than the existing literature from the presentations of S_m and S_n. The aim is achieved by considering finite groups acting on sets which lead to the construction of multiple transformations as representatives of some finite groups.

Keywords: *Cartesian product, Group action, Representation, Symmetric group, Permutation*

1. INTRODUCTION

The idea of Group arises in nature as "sets of symmetries (of an object), which are closed under composition and inverses". A good example is the Symmetric group S_n of all permutations of *n* - objects; the group of even permutations in S_n called Alternating group A_n; the Dihedral group D_{2n} (also called geometric group) which is the group of symmetries of regular *n*-gon in the plane; the Orthogonal group O(3) called the group of distance-preserving transformations in the Euclidean space that fixes the origin. From geometric point of view, questions such as "Given a geometric object *X*, what is its group of symmetries?" aroused while the same question is reversed in Representation theory such as "Given a group *G*, what objects *X* does it act on?" and the attempt to answer such question leads to the classification of *X* up to isomorphism.

In group theory, a presentation of a group G is described as a homomorphism from the group into another group, say K. It is considered as a compact way of describing the structure of any group. A representation of a group is also a presentation such that the target group is given by the group of automorphisms of a vector space. In this case, every element of the group is mapped to an invertible linear transformation in the space. The goal of group representation theory is to study groups via their actions on vector spaces. Consideration of groups acting on sets leads to such important results as the Sylow theorems. By acting on vector spaces even more detailed information about a group can be obtained. This is the subject of representation theory. It also served as a powerful tool to obtain information about finite groups with applications to many areas of sciences such as signal processing, cryptography, sound compression which is very much based on the Fast Fourier Transform (FFT) for finite groups (Knapp, 1996; Hamermesh, 1989). Its emergence was also to serve as a tool for obtaining information about finite groups through the methods of linear algebra, such as eigenvalues, inner product spaces and diagonalization.

This paper aimed at addressing a long-standing problem for creating short-length presentation for finite groups of degree *n*. An attempt by Bray *et al*. (2007), paved a way for such construction for which some short presentations for finite groups were derived. But these presentations can be made shorter with fewer relations which leads to the novelty of this paper.

1.1 PRELIMINARIES

If K is a field and G is a group, then we define a representation of G as the pair (ρ, V) where V is a vector space over K and ρ is a homomorphism of G, given by $\rho: G \to GL_K(V)$. We also defined K-algebra as a ring whose underlying Abelian group is a K-vector space such that the multiplication map $R \times R \to R$ is K-bilinear. We shall now define the following terms (see Anupam, 2011).

Definition 1.1.1: (Equivalence): Two representations: φ:G→GL(V) and ψ:G→GL(W) are equivalent if there exists an isomorphism T:V→W such that $\psi_g = T\varphi_g T^{-1}$ for all g∈G, i.e. $\psi_g T = T\varphi_g$ for all g∈G. Hence, we write $\varphi \sim \psi$.

Definition 1.1.2: (Irreducible representation): A representation $\varphi : G \to GL(V)$ is said to be irreducible if the only G-invariant subspace of V are {0} and V.

Definition 1.1.3: (Completely reducible): Let G be a group. A representation $\varphi : G \to GL(V)$ is said to be completely reducible if $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$ where the V_i are non-zero G-invariant subspaces and $\varphi|V_i$ is irreducible for all *i* = 1, 2,…,n.

Equivalently, φ is completely reducible if $\varphi \sim \varphi^{(1)} \oplus \varphi^{(2)} \oplus \ldots \oplus \varphi^{(n)}$ where the $\varphi^{(i)}$ are irreducible representations.

Definition 1.1.4: (Decomposable): We say that V is decomposable if $V = V_1 \oplus V_2$ with V_1, V_2, non-zero G-invariant subspaces. Otherwise V is called indecomposable.

Definition 1.1.5: Let (ρ_1, V_1) and (ρ_2, V_2) be representations. The linear map $T:V_1 \to V_2$ is called an intertwiner if

$T(\rho_1(g)v) = \rho_2(g)(T(v))$ or $T \circ \rho_1(g) = \rho_2(g) \circ T$ for all $g \in G$.

Lemma 1.1.6: (Shur's Lemma 1): If K is algebra closed, V is finite dimensional simple representation of G, then every self-intertwiner T:V→V is a scalar multiple of id_V.

Note: Two spaces V_1 and V_2 are isomorphic if there exists a bijective intertwiner $T:V_1 \to V_2$ and we write $V_1 \cong V_2$.

Lemma 1.1.7: (Shur's Lemma 2): If V_1 and V_2 are simple, then every non-zero intertwiner is an isomorphism. Consequently, either $V_1 \cong V_2$ or $Hom_G(V_1, V_2) = 0$.

We shall now write ϕ_g for $\phi(g)$ and $\phi_g(v)$, or simply $\phi_g v$, for the action of ϕ_g on $v \in V$.

Note: A Coxeter group *W* is defined as a group with the following presentations:

$$\left\langle x_1, x_2, \ldots, x_m \mid (x_i x_j)^{n_{ij}} = e \right\rangle$$

Where, $n_{ii} = e$ and $n_{ij} \geq 2$ for $i \neq j$ and the condition that $n_{ij} = \infty$ means there is no any relation of the form $(x_i x_j)^n$. The pair (*W*,*S*) with set of generators $S = \{x_1, \ldots, x_n\}$ is called a Coxeter system. Hence, we have the following Coxeter relations:

i. The relation $n_{ii} = e$ means that $(x_i x_i)^1 = (x_i)^2 = e$ for all *i*;

ii. If $n_{ij} = 2$, then the generators x_i and x_j commute since $aa = bb = e$ with $abab = e$ implies that $ab = a(abab)b = (aa)ba(bb) = ba$. Alternatively, the generators are involutions so that $x_I = x_i^{-1}$ and thus,

$$(x_i x_j)^2 = x_i x_j x_i x_j = x_i x_j x_i^{-1} x_j^{-1} = [x_i, x_j],$$

equal to the commutator.

iii. If redundancy among relations muost be avoided, then it is necessary to assume that $n_{ij} = n_{ji}$ by observing that $xx = e$ and $(xy)^n = e$ implies that

$$(xy)^n = (yx)^n xx = x(xy)^n x .$$

Alternatively, using conjugate elements,

$$y(xy)^m y^{-1} = (yx)^m yy^{-1} = (yx)^m$$

2. REVIEW OF RELEVANT WORK

If $\phi : Z_n \to C$ and $\varphi : Z_n \to C$ are representations on Z_n defined by $\phi_m = e^{\frac{2\pi im}{n}}$ and $\varphi_m = e^{-\frac{2\pi im}{n}}$-, respectively, then the sum $\phi \oplus \varphi$ can be define by

$$(\phi \oplus \varphi)_m = \begin{pmatrix} e^{\frac{2\pi mi}{n}} & 0 \\ 0 & e^{\frac{-2\pi mi}{n}} \end{pmatrix}.$$

Since representations are a special kind of homomorphism, if a group *G* is generated by a set *X*, then a representation *ϕ* of *G* is determined by its values on *X*; (Benjamin, 2009). If ϕ: $G \to GL(V)$ is any representation and if $W \leq V$ is a *G*-invariant subspace, then *ϕ* may be restricted to obtain a representation $\phi|_W : G \to GL(W)$ by setting $(\phi|_W)_g(w) = \phi_g(w)$ for all $w \in W$. Hence if *W* is *G*-invariant, then $\phi_g(w) \in W$ and $\phi|_W$ is sometimes called a *subrepresentation* of *ϕ*. Also, any degree one representation, say ϕ: $G \to C$ is irreducible, i.e., if *G* = {1} is the trivial group and ϕ: $G \to GL(V)$ is a representation, then $\phi_1 = e$ and if ϕ: $G \to GL(V)$ is a representation of degree 2 (i.e., *dimV* = 2), then *ϕ* is irreducible if and only if there is no common eigenvector *v* to all ϕ_g with $g \in G$ (Benjamin, 2009).

Despite the fact that numerous properties of group representations are presented in various literature, no attempt for generating and producing shorter length presentations for finite groups. In the quest to generate short presentations for finite groups, (Bray et al., (2007) derived new families of presentations based on generators and relations for the symmetric group S_n and the group of even permutations in S_n. The literature includes presentations of length that are linear in log*n* and 2-generator presentations with a bounded number of relations independent of *n*. Bray et al.The authors were able to derived the presentations for finite groups S_{m+n} with |*M*| + |*N*| + 12 relations, S_{2m} with |*M*| + 6 relations and S_{mn} with |*M*| + |*N*| + 20 relations based on the presentation of S_n as follows:

Comment [M1]: Provide example References

Theorem 2.1: Let $P = \{A \mid R\}$ and $Q = \{B \mid S\}$ be presentations for the symmetric groups S_m and S_n of degree m, n ≥ 3 respectively, such that the generating set A for Sm contains a and v standing for the transposition (1 2) and the m-cycle (1 2 … m), respectively and the generating set B for Sn contains elements b and w standing for the transposition (1 2) and the n-cycle (1 2 … n) respectively. Then

$$\{A, B, t, y \mid R, S, t^2, (at)^3, (tb)^3, y^{-1}wtv, [a,b], [a,w], [v,b], [v,w], [av,t], [vav^{-1}, t], [t, wb][t, w^{-1}bw]\}$$

is a presentation for S_{m+n} on a generating set that includes the elements y standing for the (m + n)-cycle (1 2 … m + n) and t standing for a transposition offo the form (i, i+1). This presentation has |A| + |B| + 2 generators and |R| + |S| + 12 relations, and presentation length of at most l(P) + l(Q) + 64 where l(P) and l(Q) are the lengths of the presentations P and Q (Bray et al, 2007).

Theorem 2.2: Let $P = \{A \mid R\}$ be a presentation for the symmetric group S_n of degree n ≥ 3, such that the generating set A contains x and w standing for the transposition (1 2) and the n-cycle (1 2 … n) respectively. Then

$$\{A, y \mid R, y^{2n}, (xy)^{2n-1}, [x, wy^{-1}], [w^2 xw^{-1}, wy^{-1}], [x, y^n]^2, [x, y^{n-1}]^2\}$$

is a presentation for S_{2n} on a generating set that includes the elements y standing for the 2n-cycle (1 2 … 2n) and x standing for a transposition fo the form (i, i+1). This presentation has |A| + 1 generators and |R| + 6 relations (Bray et al, 2007).

Theorem 2.3: Let $P = \{A \mid R\}$ and $Q = \{B \mid S\}$ be presentations for the symmetric groups S_m and S_n of degree m, n ≥ 3 respectively, such that the generating set A for S_m contains *a* and *v* standing for the transposition (1 2) and the m-cycle (1 2 … m) respectively and the generating set B for S_n contains elements *b* and *w* standing for the transposition (1 2) and the n-cycle (1 2 … n) respectively. Then

$$\{A, B, t, y \mid R, S, t^2, b^{-1}(v^{-1}tw^{-1}v^{-1}w)^m, w^{-1}y^m, y^{-1}wv(wtv)^{n-1}, y^{-1}vyav^{-1}t, (v^2av^{-2}t)^3, (tw^{-1}aw)^3,$$

$$[a, t], [v^2av^{-1}, t], [a, vy^{-1}], yty^{-1}v^2av^{-2}, y^{-1}tyw^{-1}aw, [a, w^{-1}aw], [a, w^{-1}vw], [v, w^{-1}aw], [v, w^{-1}vw],$$

$$[a, wb], [a, w^{-1}bw], [v, wb], [v, w^{-1}bw]\}$$

is a presentation for S_{mn} on a generating set that includes the elements *y* standing for the mn-cycle (1, 2, …, mn) and *t* standing for a transposition fo the form (i, i+1). This presentation has |A| + |B| + 2 generators and |R| + |S| + 20 relations (Bray et al, 2007).

It is observed that the generated presentations in this literature can be obtained with fewer relations. This work therefore, presents a concrete technique for generating shorter presentations for finite groups with few relations.

3. METHODOLOGY

In this section, the method of constructing presentations for the finite group S_n having length that is linear in *n* is presented as discussed by (Bray et al., (2007). But we shall first present the Cartesian product of sets $S_1, S_2, \ldots, S_n$ as the set of all ordered *n*-tuples $(x_1, x_2, \ldots, x_n)$, where $x_i \in S_i$. The Cartesian product is usually denoted by either

$$S_1 \otimes S_2 \otimes \ldots \otimes S_n \text{ or by } \prod_{i=1}^{n} S_i .$$

Now, let the binary operations on the groups $G_1, G_2, \ldots, G_n$ be multiplication. Regarding the G_i as sets, we can form the Cartesian product $\prod_{i=1}^{n} G_i$ of the groups $G_1, G_2, \ldots, G_n$. It is also easy to make $\prod_{i=1}^{n} G_i$ into a group by means of a binary operation of multiplication by components. Hence, new groups can be formed from Cartesian product of known groups as presented by the following theorems:

Theorem 3.1: (see Lang, 2002): Let $G_1, G_2, \ldots, G_n$ be groups. For $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ in $\prod_{i=1}^{n} G_i$, define $(x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n) = (x_1 y_1, x_2 y_2, \ldots, x_n y_n)$. Then $\prod_{i=1}^{n} G_i$ is a group called the External Direct Product of the groups $G_1, G_2, \ldots, G_n$ under this binary operation.

Remark 3.2: It can be deduced from the above theorem that for the groups $G_1, G_2, \ldots, G_n$ with orders $r_1, r_2, \ldots, r_n$, respectively, we have

$|G_1 \otimes G_2 \otimes \ldots \otimes G_n| = |G_1|\,|G_2| \ldots |G_n| = r_1 r_2 \ldots r_n$ where the product $G_1 \otimes G_2 \otimes \ldots \otimes G_n$ is a new group which may or may not be isomorphic to the group $G_{r_1 r_2 \ldots r_n}$.

Theorem 3.3: (see Lang, 2002): The group $Z_m \otimes Z_n$ is isomorphic to Z_{mn} if and only if $(m, n) = 1$.

Now, let G = S_n whose elements are bijections on the set S. Then to obtain a presentation for G, we introduced an *n*-cycle ξ = (1, 2, …, *n*) to be a new generator and then used the fact that $\xi^{-j}\alpha_i\xi^j = (j, j+1) = \alpha_{j+1}$ to eliminate the generator α_i for $1 \le j < n$. If we take an arbitrary generator α and then eliminate further redundancy of relations under conjugation by ξ, then the presentation is given by

$$\{\alpha, \xi \mid \alpha^2 = \xi^n = (\alpha\xi)^2 = e, (\alpha\xi^{-1}\alpha\xi)^3 = e, (\alpha\xi^{-j}\alpha\xi^j)^2 = e;\ j = 2, \ldots, n/2\}.$$

However, if we define $\xi_j = \xi^j$, then $(\alpha\xi^{-j}\alpha\xi^j)^2 = e$ is replaced by $(\alpha\xi_j^{-1}\alpha\xi_j)^2 = e$. Hence, we have the following:

Theorem 3.4: (Bray et al, 2007): For all $n \ge 3$, the symmetric group S_n has the following presentation:

$$\{\alpha, \xi_1, \ldots, \xi_{n/2} \mid \alpha^2 = \xi_1^n = (\alpha\xi_1)^{n-1} = e, (\alpha\xi_1^{-1}\alpha\xi_1)^3 = e, \xi_{j-1}\xi_1\xi_j^{-1} = e, (\alpha\xi_j^{-1}\alpha\xi_j)^2 = e\}$$

with 1 + *n*/2 generators and *n* + 2 relations.

Again, let $S_n = \langle \sigma_i \mid 0 \le i \le n-1 \rangle$ where σ is any bijection from 1 to *n* such that $\sigma_0 = \sigma_n = e,$ the identity element of G. Then we shall have the following relations:

If $\sigma_i = (i, i+1),$ then

Relation 1: $(\sigma_i)^2 = e;$

Relation 2: for all *i*, $(\sigma_i\sigma_{i+1})^3 = e$; $(\sigma_i\sigma_{i+1} = (i, i+2, i+1), (\sigma_i\sigma_{i+1})^{-1} = (i, i+1, i+2))$;

Relation 3: for all *i*, *j*, $|i - j| \ge 2$, $(\sigma_i\sigma_j)^2 = e;$

So that if $P = \langle \sigma_i^2 \mid 0 \le i \le n-1 \rangle,$ $Q = \langle (\sigma_i\sigma_j)^2, i < j, |\ j - i| > i \rangle$ and $R = \langle (\sigma_i\sigma_{i+1})^3 \mid 0 \le i \le n-1 \rangle,$ taking *M* as a finite group such that $M = P \cup Q \cup R,$ then any finite group $G_n = \langle X \mid M \rangle,$ $X = \{\sigma_i \mid 1 \le i \le n\}$ is isomorphic to S_n.

From the methods presented above, the presentations for S_{m+n}, S_{2n} and S_{mn} with less relations are obtained in the next Section.

4. RESULT AND DISCUSSION

Following the methodology above, we present in this section the key idea for obtaining short presentations for the finite groups S_{m+n} and S_{mn} for all $m,n \in Z^+$ from the presentations of S_m and S_n. When *m* = *n*, we avoid repeating the presentation for S_n and this enable us to efficiently construct a shorter presentation for S_{2m} from the presentation of S_n. Hence, an inductive process for obtaining a presentation for S_n for all $n \in Z^+$ is achieved.

Theorem 4.1: Let $S_m = \{X | M\}$ and $S_n = \{Y | N\}$ be presentations for S_m and S_n with generating sets *X* and *Y* respectively, where *M* and *N* denote the set of relations for S_m and S_n. Let τ, δ be transpositions and ϕ, φ be rotations through $\frac{2\pi}{k}$ rad such that $\tau, \phi \in X$ and $\delta, \varphi \in Y$. Then the presentation for S_r where $r = m+n$, is given by

$$\{X, Y, \nu, \omega \mid M, N, \nu^2, (\tau\nu)^3, (\delta\nu)^3, [\tau,\varphi], [\tau,\delta], [\phi,\delta], [\phi,\varphi], [\phi\tau\phi^{-1},\nu], [\nu,\varphi^{-1}\delta\varphi], \omega^{-1}\varphi\nu\phi\}$$

whereWhere, ω represents the m+n – cycle (1, 2, …, m+n) and v a transposition of the form $(i, i+1)$. The given presentation has |X| + |Y| + 2 generators and |M| + |N| + 10 relations.

Proof: Let G be the group defined by the given presentation. Define $\tau,\delta,\phi,\varphi \in$ G by

$$\tau = \omega\nu\omega^{-1},\ \ \phi = (\omega\nu)^i\omega^{-i},\ \ \delta = \omega^{-1}\nu\omega,\ \ \varphi = \omega^{-j}(\nu\omega)^j$$

for all i = 1, 2, …, m-1 and j = 1, 2, …, n-1. Then the presentation is transformed into a 2-generator presentation in terms of ν and ω subject to at most |M| + |N| + 10 relations. Now, defined a homomorphism $\xi : G \to S_r$ from G to S_r such that $r = m + n$ and

$\xi(\tau) = (i, i+1)$ for all transpositions $\tau \in G$; $\xi(\phi) = (1,2,\ldots,m)$;

$\xi(\varphi) = (m+1, m+2, \ldots, m+n)$; $\xi(\omega) = (1,2,\ldots,m,m+1,\ldots,m+n)$.

Then the permutations satisfy the above relations in G. Again, if we let $\nu_{m-1} = \tau$, $\nu_{m+i-1} = \phi^i\tau\phi^{-i}$, $\nu_{m+1} = \delta$, $\nu_{m+j+1} = \varphi^{-j}\delta\varphi^j$ for all $1 \le i < m$ and $1 \le j < m$, then these relations satisfy the Coxeter relations on the group S_r and generate G. Again from the relations 1 to 3 (Section 3), if $\sigma_i\sigma_j = \tau$, then $\sigma_j\sigma_i = \tau^{-1}$. Thus,

$$[\phi\tau\phi^{-1},\nu] = (\phi\tau\phi^{-1})^{-1}\nu^{-1}\phi\tau\phi^{-1}\nu = \phi\tau^{-1}\phi^{-1}\nu^{-1}\phi\tau\phi^{-1}\nu = \phi\tau\phi^{-1}\nu\phi\tau\phi^{-1}\nu$$

$$= (\tau\phi^{-1})^{-1}\phi^{-1}\nu\phi(\tau\phi^{-1})\nu$$

$$= \alpha^{-1}\nu\alpha\nu \text{ where } \alpha = \tau\phi^{-1} \text{ and}$$

$$[\tau\phi,\nu] = (\tau\phi)^{-1}\nu^{-1}\tau\phi\nu = \phi^{-1}\tau^{-1}\nu^{-1}\tau\phi\nu = \phi^{-1}\tau\nu\tau\phi\nu$$

$$= (\tau\phi)^{-1}\nu(\tau\phi)\nu$$

$$= \beta^{-1}\nu\beta\nu \text{ where, } \beta = \tau\phi.$$

But if H and K are subgroups of G such that $H = \langle\tau\phi^{-1}\rangle$ and $K = \langle\tau\phi\rangle$, then obviously, $H \cong K$.

Similarly, $[\nu,\varphi^{-1}\delta\varphi] = \eta\nu\eta^{-1}\nu$ where $\eta = \varphi^{-1}\delta$ and $[\nu,\varphi\delta] = \xi\nu\xi^{-1}\nu$ where, $\xi = \varphi\delta$ so that if $M = \langle\varphi^{-1}\delta\rangle$ and $N = \langle\varphi\delta\rangle$, then $M \cong N$.

Furthermore, by hypothesis, the subgroup $K = \langle\nu_i\rangle$ is isomorphic to S_m and satisfiesy the Coxeter relation and similarly the subgroup $L = \langle\nu_{m+i}\rangle$ is isomorphic to S_n and satisfiesy the Coxeter relation. Thus, the element ν_i for $1 \le i < m+n$ satisfiesy the Coxeter relations and since $(\tau\nu)^3 = (\delta\nu)^3 = e$, we have $(\nu_i\nu_{i+1})^3 = e$ for all $1 \le i \le m+n-1$. The relation $[\nu_i,\nu_j] = (\nu_i,\nu_j)^2 = e$ holds for i<j<m from the presentation of S_m and holds for m<i<j from the presentation of S_n and if $i<m<j$, then it follows from the relations $[\tau,\delta]=[\tau,\varphi]=[\phi,\delta]=e$. Similarly, since $\tau\phi$ and $\phi\tau\phi^{-1}$ generate a subgroup K of index m in $\langle\tau,\phi\rangle\cong S_m$ which contain the involutions $v_1, v_2, \ldots, v_m$ and $\varphi\delta$ and $\varphi^{-1}\delta\varphi$ generate a subgroup L of index n in $\langle\delta,\varphi\rangle\cong S_n$ which contains the involutions $v_{m+1}, v_{m+2}, \ldots, v_{m+n}$, the relation $[\tau\phi, v]=[\phi\tau\phi^{-1}, v]=[v,\varphi\delta]=[v,\varphi^{-1}\delta\varphi]=e$ implies that the element v centralizes $\langle v_1, v_2, \ldots, v_m\rangle$ and $\langle v_{m+1}, v_{m+2}, \ldots, v_{m+n}\rangle$ so that $[v_i, v_m]=e$ and $[v_m, v_j]=e$ for all $1\le i\le m$ and $m\le j\le m+n$.

Hence, the involution $v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_{m+n}$ generates a subgroup that satisf~~y~~ies the Coxeter relations for S_r. But the relations in S_m (and S_n) implies that each of its elements can be expressed as a word in $v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_{m+n}$ and the relation $\omega^{-1}\varphi v\phi = e$ imposed the same condition for ω. Thus the same involution generates G. Hence, $G\cong S_r$ and the result follows.

Next, we consider the case $m = n$ such that $S_r = S_{m+n} = S_{2m}$.

Corollary 4.2: Let $S_m = \{X|M\}$ be a presentation for S_m, $m\ge 3$ and let $\tau,\alpha\in X$ such that $\tau = (i, i+1)$ and $\alpha = (1, 2, \ldots, m)$. Then

$$\{X,\omega \mid M,\omega^r,(\tau\omega)^{r-1},[\tau,\omega\alpha],[\alpha^i\tau\alpha^{-i},\omega\alpha],[\tau,\omega^m]^2\}$$

is the representation for S_r where $r = 2m$, $1 \le i \le m$ and a generating set that includes $\omega = (1,2,\ldots,r)$, $|X| + 1$ generators and $|M| + 5$ relations.

Proof: This follows directly from Theorem ~~4.2.1~~4.1 above with $m = n$ and the fact that if $\lambda = \tau w^{-1}$ and $\pi = \tau w$, then

$[\tau, w] = \tau^{-1}w^{-1}\tau w = \tau w^{-1}\tau w = \lambda\pi$, $[\tau, w]^2 = (\tau^{-1}w^{-1}\tau w)(\tau^{-1}w^{-1}\tau w) = \tau w^{-1}\tau w\,\tau w^{-1}\tau w = (\lambda\pi)^2$ and so on, for all w^i.

The next result is derived from Cartesian product of two groups such that given two groups H and K, then the product HK is given by the set

$$HK = \{x = hk : h\in H, k\in K\}.$$

Theorem 4.3: Let $S_m = \{X|M\}$ and $S_n = \{Y|N\}$ be presentations for S_m and S_n, $m,n\ge 3$ with generating sets X and Y, respectively, and where M and N denote the set of relations for S_m and S_n. Let τ, δ be transpositions and ϕ, φ be rotations through $\frac{2\pi}{k}$ rad such that $\tau,\phi\in X$ and $\delta,\varphi\in Y$. Then the presentation for S_{mn} is given by

$$\{X,Y,v,\omega \,|\, M,N,v^2,\delta^{-1}(\phi v\varphi^{-1}\phi^{-1}\varphi)^i,\omega^{-1}\phi\omega\tau\phi^{-1}v,(\phi^2\tau\phi^{-2}v)^3,(v\varphi^{-1}\tau\varphi)^3,[\tau,v],[\phi^2\tau\phi^{-2},v],$$
$$[\tau,\phi\omega^{-1}],\omega v\omega^{-1}\phi^2\tau\phi^{-2},\omega^{-1}v\omega\varphi^{-1}\tau\varphi,[\tau,\varphi^{-1}\tau\varphi],[\tau,\varphi^{-1}\phi\varphi],[\phi,\varphi^{-1}\tau\varphi],[\phi,\varphi^{-1}\phi\varphi][\tau,\varphi\delta],$$
$$[\tau,\varphi^{-1}\delta\varphi],[\phi,\varphi\delta],[\phi,\varphi^{-1}\delta\varphi]\}$$

Where, ω represents the mn – cycle (1, 2, …, mn) and v a transposition of the form $(i, i+1)$. The given presentation has $|X| + |Y| + 2$ generators and $|M| + |N| + 18$ relations.

Proof: Suppose *G* is the group defined by the given presentation, define a function $\xi : G \to S_{mn}$ from *G* to S_r such that $\tau \mapsto (i, i+1)$, $\phi \mapsto (1,2,\ldots,m)$, $\delta \mapsto (j, j+1)$, $\varphi \mapsto (1, 1+m, \ldots, 1+(n-1)m)(2, 2+m, \ldots, 2+(n-1)m)\ldots(m, 2m, \ldots, nm)$, $v \mapsto (k, k+1)$, and $\omega \mapsto (1,2,\ldots,m,m+1,m+2,\ldots,2m,2m+1,\ldots,mn)$.

Then ξ is a homomorphism and for some *m*, *n*, $w^m = \varphi$ and $w^n = \phi$. In particular, $\xi(\tau)$ and $\xi(\phi)$ generate a subgroup *H* of S_m such that $H \cong S_m$ and the conjugate of *H* under powers of φ generates the direct product of *n*-copies of S_m. Now, define $v_1 = \tau$, $v_{i+1} = \phi^{-i}\tau\phi^{i}$, $\lambda_{i+1} = \omega^{-i}\tau\omega^{i}$ and $v_{im+j} = \varphi^{-i} v_j \varphi^{i}$ for $1 \le i \le m$ and $1 \le j \le n$ in *G*. Then we shall show that the (*mn* – 1) elements $v_1, v_2, \ldots, v_q$, *q*=*mn* – 1 satisfy the usual Coxeter relations for S_{mn} and generate the group *G*. To see this, note that $\phi v \varphi^{-1}\phi^{-1}\varphi = \phi v(\phi\varphi)^{-1}\varphi = \phi v \sigma^{-1}\varphi = \phi\xi\varphi$ where $\sigma = \phi\varphi$ is a *mn* – cycle and $\xi = v\sigma^{-1}$ is a (*mn* – 1) – cycle, $(\phi v \varphi^{-1}\phi^{-1}\varphi)^2 = (\phi\xi\varphi)^2 = \phi\xi\mu\xi\varphi$ where, $\mu = \varphi\phi$ is a *mn* – cycle. Thus, both the product $w^{-1}\varphi\phi(\varphi v\phi)^2 = w^{-1}\varphi\sigma v\sigma v\phi = w^{-1}\varphi(\xi^{-1})^2\phi = w^{-1}\varphi\pi\phi$, $\pi = (\xi^{-1})^2$ which is a (*mn* – 1) – cycle, and $w^{-1}vw\varphi^{-1}\tau\varphi = w^{-1}\lambda_i\varphi^{-1}\lambda_j$ are *mn* – cycles.

Now, by the hypothesis on S_m, the elements $v_1, v_2, \ldots, v_m$ and $v_{im+1}, v_{im+2}, \ldots, v_{im+q}$, respectively generate the subgroups *H* and *K* of S_m such that $H \cong S_m$ and $K \cong S_m$ for $1 \le i < n$. Again, the commutator relations

$$[\tau, \varphi^{-1}\tau\varphi] = [\tau, \varphi^{-1}\phi\varphi] = [\phi, \varphi^{-1}\tau\varphi] = [\phi, \varphi^{-1}\phi\varphi] = e$$

describe the subgroup *H* as $H = \langle v_1, v_2, \ldots, v_m \rangle = \langle \tau, \phi \rangle$ which commutes with its conjugate $H_1 = \langle v_{m+1}, v_{m+2}, \ldots, v_{2m} \rangle$ under φ. The relations

$$[\tau, \varphi\delta] = [\tau, \varphi^{-1}\delta\varphi] = [\phi, \varphi\delta] = [\phi, \varphi^{-1}\delta\varphi] = e$$

implies that the subgroup *H* is centralized by the set $N_1 = \langle \varphi\delta, \varphi^{-1}\delta\varphi \rangle$ such that

$$[\langle \delta, \varphi \rangle : \langle \varphi\delta, \varphi^{-1}\delta\varphi \rangle] = n \text{ and } N_2 = \langle \delta, \varphi \rangle \cong S_n .$$

Hence, if $N_i = \{H \subseteq S_m : H \text{ is a subgroup}\}$, then N_2 permutes the subgroups N_i by conjugation according to the natural action of S_n on the index set {1,2,…,*n*}.

Next, the transposition *v* satisfies the relation $(v_i)^2 = e$ for all *i* and the relations $(\phi^2\tau\phi^{-2}v)^3 = (v\varphi\tau\varphi^{-1}v)^3 = e$ implies that $(v_i v_{i+1})^3 = e$ for all $1 \le i < m-1$ and then conjugation by powers of φ gives all the remaining relations. Again, to see that $[v_i, v_j] = (v_i v_j)^2 = e$ for $1 \le i \le j \le mn$, we first consider the presentation for S_m. If $i < j < m$, then the result follows directly from the presentation for S_m and also conjugation by φ^{i} gives the same result for $km < i < j < (k+1)m$ for some $k \in \mathbb{Z}^{+}$. Also, $[v_i, v_j] = e$ is true if both v_i and v_j lie in different conjugate sets of the subgroup H since the conjugates commute with each other. And the relations $[\tau, v] = [\phi\tau\phi^{-1}, v] = e$ ensure that v_i commutes with all the elements in $\langle \tau, \phi\tau\phi^{-1} \rangle = \langle v_1, v_2, \ldots, v_m \rangle$.

The rest of the relations will follow if $[\tau, \phi\omega^{-1}] = \omega^{-1}\phi\omega\tau\phi^{-1}v = \omega v \omega^{-1}\phi^{2}\tau\phi^{-2} = \omega^{-1}v\omega\varphi^{-1}\tau\varphi = e$ giving~~es~~ the conjugation of ω on each successive pair of the elements in $\{v_1, v_2, ..., v_{mn}\}$. Thus, since $\phi\omega^{-1}$ centralizes τ, we have

$$\omega v_1 \omega^{-1} = \omega\tau\omega^{-1} = \phi\tau\phi^{-1} = v_2$$

~~and~~And we find by induction on *i*, for $1 \le i \le m-2$, that

$$\omega v_{i+1}\omega^{-1} = \omega\phi v_i \phi^{-1}\omega^{-1} = (\omega\phi\omega^{-1})^{-1}(\omega v_i \omega^{-1})(\omega\phi\omega^{-1})$$
$$= (\tau\phi^{-1}v)v_{i+1}(v\phi\tau) = \tau\phi^{-1}v_{i+1}\phi\tau^{-1} = \tau v_{i+2}\tau^{-1} = v_{i+2}$$

~~since~~Since *v* commutes with each v_i, $\tau = v_1$ commutes with v_{i+2}. Also,

$$\omega v_{m-1}\omega^{-1} = \omega\phi^{-2}\tau\phi^{2}\omega^{-1} = v_m \text{, and } \omega v_m \omega^{-1} = \omega v \omega^{-1} = \varphi\tau\varphi^{-1} = \varphi v_1 \varphi^{-1} = v_{m+1}$$

~~and~~And since ω centralizes $\varphi = \omega^m$, we find that

$$\omega v_{im+j}\omega^{-1} = \omega\varphi^{i}v_j\varphi^{-i}\omega^{-1} = \varphi^{i}\omega v_j \omega^{-1}\varphi^{-i} = \varphi^{i}v_{j+1}\varphi^{-i} = v_{im+j+1}$$

~~for~~For $1 \le i \le n$ and $1 \le j \le m$.

Again, conjugation by powers of ω satisfies~~y~~ all the relations of the form $[v_i, v_j] = e$ for $1 \le i < j \le mn$. Thus, the $mn-1$ involutions $\{v_1, v_2, ..., v_{mn-1}\}$ generate a subgroup that satisfies~~y~~ the usual Coxeter relations for S_{mn}.

Finally, to show that each generator of G can be expressed as a word in v_i, we first consider the relations in S_m. Obviously, the set M satisfies~~y~~ this condition for each element in the set *X*. In particular, $\tau = v_1$ and

$$\phi = (\phi^{m-2}\tau\phi^{-(m-2)})(\phi^{m-3}\tau\phi^{-(m-3)})...(\phi\tau\phi^{-1})\tau = v_{m-1}v_{m-2}...v_1$$

which follows that

$$\varphi^{i}\phi\varphi^{-i} = v_{im+m-1}v_{im+m-2}...v_{im+2}v_{im+1}$$

~~for~~For $1 \le i < n$ and similarly, $v = v_m$ and $\varphi^{i}v\varphi^{-i} = v_{(i+1)m}$ for $1 \le i \le n-2$. Hence, we deduced from $\omega = \varphi\phi(\varphi v\phi)^{n-1}$ and $\varphi^{i}\phi\varphi^{-i} = v_{im+m-1}v_{im+m-2}...v_{im+2}v_{im+1}$ that

$$\omega = (v_{mn-1}...v_{m(n-1)+1})v_{m(n-1)}...(v_{2m-1}...v_{m+2}v_{m+1})v_m(v_{m-1}...v_2v_1) = v_{mn-1}v_{mn-2}...v_2v_1$$

~~and~~And from the relations $\varphi = \omega^m$ and $\delta = (\varphi\phi^{-1}\varphi^{-1}v\phi^{-1})^m$, it follows that both elements δ and φ are expressible as words in the set v_i. Hence, the involutions $\{v_1, v_2, ..., v_{mn}\}$ generate *G* so that $G \cong S_{mn}$ and the result follows.

5. CONCLUSION

This work presents new families of presentations by generators and relations. The result gives shorter presentations for the symmetric groups S_{m+n}, S_{2n} and S_{mn} with |M| + |N| + 10 relations, |M| + 5 relations and |M| + |N| + 18 relations, respectively (Theorem 4.1, Corollary 4.2 and Theorem 4.3). For demonstration purpose, with $n \ge 3$, $X = \{\sigma, \tau\}$, $Y = \{\sigma, \tau, \omega\}$ and $Z = \{\sigma, \tau, \omega, \lambda\}$, we have:

$$S_3 \cong \langle X : \sigma^2 = \tau^3 = (\sigma\tau)^2 = e \rangle;$$

$$S_4 \cong \langle X : \sigma^2 = \tau^4 = (\sigma\tau)^3 = e \rangle;$$

$$S_5 \cong \langle Y : \sigma^2 = \tau^5 = \omega^3 = e, (\sigma\omega)^2 = \tau^{-1}\sigma\tau^2\sigma\omega = e \rangle;$$

$$S_6 \cong \langle Z : \sigma^2 = \omega^5 = e, \tau\omega^2\lambda^{-1} = \tau\lambda^{-1}\tau^{-1}\omega\lambda = \sigma\lambda\tau^{-1}\omega^{-1}\lambda = e \rangle;$$

$$S_7 \cong \langle Z : \sigma^2 = e, \tau^{-2}\omega\lambda^2 = \tau^{-1}\omega\lambda^{-1}\omega\lambda = \sigma\tau^{-1}\omega^{-1}\lambda\omega\tau^{-1} = (\sigma\tau^{-1}\lambda^{-1})^2 = e \rangle;$$ and

$$S_8 \cong \langle Z : \sigma^2 = \omega^2 = (\sigma\omega)^2 = e, \lambda^{-1}\tau^{-1}\sigma\tau\lambda\sigma = \lambda^{-1}\omega\sigma\tau^{-1}\lambda^{-1}\omega = \tau^4\lambda^{-1}\omega\lambda = e \rangle;$$

Group representation theory states that new representations can be constructed by direct product or tensor product of any two representations. Their irreducible representation is also a direct product or tensor product. This work clearly presents a shorter and simpler method for building presentations for the finite groups S_{m+n}, S_{2n} and S_{mn} from the representations of S_m and S_n with less number of relations than the existing literature.

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