Analytical solution of transport equation for exponentially decreasing initial concentration in shallow water table condition in irrigated field

Abstract

Most of the agricultural activities is limited for the depth of 15 – 20 cm and rest soil is remains enact for a long periods which inhibits the microbial activities below this depths and create a initial concentration of nutrients exponentially decreasing with depth. Attempt has been made to develop analytical models for time dependent nitrification/ denitrification and depth dependent absorption of urea fertilizer in high water table conditions with fertigation. Laplace transformation method was used to solve the unsteady-state advection-dispersion equation. The analytical solutions that can be derived by this method assist understanding of movement of fertilizer in irrigated fields. The developed models were validated with the experimental results. They were closely predicting the fertilizer movement in one-dimension soil medium. The little deviation of result from observed values may be due to change of dispersion coefficient and velocity with moisture content. Here these parameters were assumed as constant throughout the time under consideration. Models developed for constant degradation rate is predicting very close to observe values which shows that the soil under study have no depth dependent degradation. The developed models may be helpful for planning of drain design, nutrient management and assessment of potential hazards to groundwater in agricultural fields by the knowledge of exact transport parameters and boundary conditions universally.

Comment [DL1]: This is not true, it is always between 0 and 30 cm oe where is the literature to back this up?

Comment [DL2]: this depth?

Introduction

The top soil up to a depth of 15-20 cm supports the most of the seasonal and shallow deep rooted cereals crops. These types of crops may be grown successfully without tillage in a wide range of soil. Popularity of zero-till machines in Indian agriculture avoids the tillage of soil up to a great extent. Addition of different biofertilizers and soil ameliorators increase the biological activities in upper soil and leave the lower soil enact and creates the situation of exponentially decreasing initial concentration of different soil nutrients in soil mass in high water table areas. Mathematical modelling of processes, in the unsaturated zone, is useful for the agricultural management of cultivated sites, for prediction of the fate of agrochemicals, and for assessment of the potential hazard of shallow groundwater contamination. The difficulty of solving the transport equation in the unsaturated zone relies on its strong nonlinearity. Although significant efforts have been made to overcome the mathematical difficulties, most analytical solutions are derived for one-dimensional vertical transport under various simplifying assumptions. Accounting for the spatial heterogeneity of natural soils renders the transport problem even more complicated.

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Comment [DL4]: Example in literature form?

Bresler and Laufer (1974) simulated the movement of nitrate in homogeneous soil profile in the presence of NO_3-N production (nitrification). *Sexton et al.* (1977) modelled nitrate - nitrogen movement and dissipation in fertilized agricultural lands, but did not include representation of any other fertilizer from nitrogen. *Wagenet et al.* (1977) extended the mathematical analysis of *Cho (1971*) and *Misra et al*. (1974) to describe the transport and transformation of urea, ammonium nitrogen and nitrate nitrogen soil profile as a function of depth and time subject to either steady or pulse feed application of nitrogen, and validated with controlled laboratory experiments.

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Davidson et al. (1978) developed research and development type models on the fate of nitrogen in the root zone by simplifying assumptions of water and solute processes in the field. *Watts and Hanks* (1978), *Tillotson et al* (1980), *Tillotson and Wagenet* (1982) developed a model that simulated most of the major transformations of the nitrate as well as the uptake by the crop, but fell short of fully describing the system in the plant growth and yield response. *Selim and Iskander* (1981) developed the model for calculating pollution from organic wastes and from excessive fertilization. *Tanji et al*. (1982) presented a steady state nitrogen model developed on a mass balance approach, which considered water and nitrogen flow on annual or cropped cycle time basis. *Barraclough (1989), Borg et al. (1990), Benbi et al. (1991), Carbon et al. (1991*) also developed the soil water nitrogen models. *Izadi et al*. (1996) combined functional sub model and analytical solution to the steady state convection dispersion equations to predict the nitrate concentration in the root zone.*Lessoff and Indelman* (2004) investigated the penetration of reactive solute into a soil during a cycle of water infiltration and redistribution by deriving analytical closed form solutions for fluid flux, moisture content and contaminant concentration *Sander* and *Braddock* (2005) presented a range of analytical solutions to the combined transient water and solute transport for horizontal flow. Smedt (2007) reported an analytical solution and analysis of solute transport in rivers affected by diffusive transfer in the hyporheic zone. Khakpour and Kambiz (2008) reported analytical solution of transport phenomena within PEM fuel cell. Zhan et al (2009) deduced an analytical solution of two-dimensional solute transport in an aquifer–aquitard system. Srinivasan and Clement (2008) reported analytical solutions for sequentially coupled one-dimensional reactive transport problems. Sadek (2009) compared various available analytical solution with numerical methods is deduced that analytical solution may be use as a versatile tool for assessment of contaminant transport. Jozse and Janos (2009) derived analytical solution of Analytical solution of the coupled 2-D turbulent heat and vapor transport equations and the complementary relationship of evaporation . Guerrero and Skaggs (2010) presented a general analytical solution for linear, one dimensional advection dispersion equation with distance dependent coefficients. An integrating factor was employed to obtain a transport equation that has a self-adjoint differential operator, and a solution was found using the generalized integral transform technique.

The mechanisms of solute transport in irrigated field are significantly influenced by attenuation processes such as adsorption and nitrification/denitrification processes. Most of the available analytical solutions are based on linear equilibrium adsorption and first order nitrification and possibly zeroth order production (*van Genuchten* and *Alves* (1982) for a number of analytical solutions). Here movement of urea fertilizer was analytically solved under depth dependent adsorption factor and combination of constant and exponential nitrification/denitrification rate for exponentially decreasing initial condition. Following assumptions were considered for formulating the boundary value problems:

1. The soil is unconfined, homogeneous and isotropic overlying an impermeable layer which is having water table depth H meter from soil surface,
2. The water through deep percolation moves vertically downward until it joins the ground water,
3. Darcy and Fick's laws hold good,

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4. Fluid is of constant density and viscosity,

In the present study 1-D Richard's equation in combination with solute transport equation, which incorporates nitrification and de-nitrification, and depth dependent soil and water matrix factor was used to characterize the movement of applied fertilizer in irrigated agriculture having shallow water table conditions.

Governing Equation

Transport equation in unsaturated porous medium is given by:

$$\frac{\partial}{\partial t}(\theta C+\rho S)=\frac{\partial}{\partial z}\left(\theta D\frac{\partial C}{\partial z}-qC\right)-\alpha\theta C-\rho\beta C\mp\gamma\theta \qquad \ldots(1)$$

where C = C(z, t) is the concentration of chemical in the liquid phase in mg/l , S = S (z, t) is the concentration of chemical in the solid phase in mg/l, D = D(z, t) is the dispersion coefficient in m^2/day, $\theta = \theta$ (z, t) is the volumetric water content cm^3 /cm^3, q = q(z, t) is the flux of water in m/day, $\rho = \rho$ (z) is the soil bulk density in gm/cm^3, $\alpha = \alpha$ (z) is the first-order degradation rate constant in the liquid phase, $\beta = \beta$ (z) is the first-order degradation rate constant in the solid phase, $\gamma = \gamma$ (z) is the zero-order production rate constant in the liquid phase. Here α, β and γ are zero or greater.

Considering that soil medium remains intact with time, and introducing mass balance equation for one dimensional unsteady unsaturated flow condition as given by Chow *et al*.(1988), Eqn. (1) reduces to:

$$\frac{\partial C}{\partial t}=D\frac{\partial^2 C}{\partial z^2}-v\frac{\partial C}{\partial z}-RC\mp\gamma \qquad \ldots(2)$$

where $v=q/\theta$ and $R=\left(\alpha+\frac{\rho\beta}{\theta}\right)$, the factor representing the combined effect of liquid and solid phase degradation rate. Here we assume R(C) = R_o – b (C_o - C) where R_o represents potential degradation rate at the land surface; b is reduction factor due to which degradation decreases linearly as the depth from the land surface it increases upto a specific value; and C_o is initial concentration at the ground surface. For the development of model, combination of constant and exponentially decreasing nitrification/de-nitrification rate which may be given as $\gamma(t)=\gamma_0+\gamma_1 e^{-rt}$, where γ_0 and γ_1 are constant nitrification/de-nitrification rates, r is decay constant and t represents time.

The initial and boundary conditions in mathematical terms, for the solute flow problem in unsaturated zone under above situation, may be written as:

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$$\left.\begin{array}{lll} C(z,o)=C_z(z,o) & \text{at } t=o & \text{for } o<z<H \\ C(o,t)=C_1 & \text{at } t>o & \text{for } z=o \\ C(H,t)=C_2 & \text{at } t>o & \text{for } z=H \end{array}\right\} \quad \ldots(3)$$

where, $C_1 = g_1e^{-h\,t}$, $C_2 = g\,e^{h\,t}$, g_1 and g are the concentrations at ground surface and H meter below the soil surface before application of fertigation. C_z (z,0) is the distribution of initial concentration in the porous medium. Devising a transform given by Eqn.(4) converted the Eqn. (2) and Eqn.(3) into standard heat flow equation and given by Eqn (5) and Eqn(6), respectively.

$$C(z,t) = V(z,t)\exp\left(\frac{v\,z}{2D} - \left(\frac{v^2}{4D} + b\right)t\right) + \frac{\gamma_1 e^{-r\,t}}{b-r} + \frac{\gamma_0 + bC_0 - R_0}{b} \quad \ldots(4)$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{1}{D}\frac{\partial V}{\partial t} \quad \ldots(5)$$

$$\left.\begin{array}{l} V(z,0) = \left(C_z - A - B\right)e^{-a\,z} = f(z) \\ V(0,t) = \left(C_1 - A - B\,e^{-r\,t}\right)e^{d\,t} = f_1(t) \\ V(H,t) = E\left(C_2 - A - B\,e^{-r\,t}\right)e^{d\,t} = f_2(t) \end{array}\right\} \quad \ldots(6)$$

where, $A = \dfrac{\gamma_0 + bC_0 - R_0}{b}$, $d = \dfrac{v^2}{4D} + b$, $a = \dfrac{v}{2D}$, $E = e^{-aH}$ and $B = \dfrac{\gamma_1}{b-r}$

The general solution of transformed Eqn (5) under initial and boundary condition Eqn(6) is given by *Carslaw* and Jaeger (1959) and *Ozisik* (1980) as below:

$$V(z,t) = \frac{2}{H}\sum_{m=1}^{\infty} e^{-D\beta_m^2 t}\sin\beta_m z\left[\int_0^H f(z)\sin\beta_m z\,dz\right] + \left(1 - \frac{z}{H}\right)f_1(t) + \frac{z}{H}f_2(t)$$

$$-\frac{2}{H}\sum_{m=1}^{\infty}\frac{\sin\beta_m z}{\beta_m}\left[f_1(0)\,e^{-D\beta_m^2 t} + \int_0^t e^{-D\beta_m^2(t+\tau)}df_1(\tau)\right]$$

$$+\frac{2}{H}\sum_{m=1}^{\infty}(-1)^m\frac{\sin\beta_m z}{\beta_m}\left[f_2(0)\,e^{-D\beta_m^2 t} + \int_0^t e^{-D\beta_m^2(t+\tau)}df_2(\tau)\right] \quad \ldots(7)$$

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where, β_m is the root of $\sin\beta_m$ H = 0 and τ is a dummy variable. Solution of transport equation was obtained for exponentially decreasing initial concentration of nitrogen in soil profile with the help of equation (7) and transformed initial and boundary conditions. When $C_z = p - e^{kz}$, i.e**.** exponentially decreasing with depth, then final solution of Eqn. (7) takes the following form:

$$C(z,t) = \frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} A_2 \sin\beta_m z \left[(p-A-B) \frac{\beta_m}{a^2+\beta_m^2} \left[1-(-1)^m E\right] \right]$$

$$-\frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} A_2 \sin\beta_m z \ \frac{\beta_m}{(k-a)^2+\beta_m^2} \left[1-(-1)^m E e^{kH}\right]$$

$$+\left(1-\frac{z}{H}\right)\left(C_1 - A - Be^{-rt}\right)e^{az} + \frac{z}{H}\left(C_2 - A - Be^{-rt}\right)e^{a(z-H)}$$

$$-\frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} \frac{\sin\beta_m z}{\beta_m} \left[(g_1 - A - B)A_2\right]$$

$$-\frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} A_2 \frac{\sin\beta_m z}{\beta_m} \left[\frac{A_6 A_2}{s}\left(e^{-st}-1\right) \frac{A_4 A_2}{K}\left(e^{-Kt}-1\right) - \frac{A_2 B_1}{N}\left(e^{-Nt}-1\right) \right]$$

$$+\frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} (-1)^m A_2 \frac{\sin\beta_m z}{\beta_m} \left[(g-A-B)E\right] + A + Be^{-rt}$$

$$+\frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} (-1)^m A_2 E \frac{\sin\beta_m z}{\beta_m} \ \frac{A_3}{l}\left(e^{-lt}-1\right) - \frac{B_1}{N}\left(e^{-Nt}-1\right) - \frac{A_4}{K}\left(e^{-Kt}-1\right) \qquad \ldots(8)$$

$A_2 = e^{-D\beta_m^2 t}$ $\quad$ $A_3 = (h+d)g$ $\quad$ $A_4 = d\,A\,A_5 = (C_0 - A - B)$

$A_5 = (C_0 - A - B)$ $\quad$ $A_6 = g_1(d-h)$, $\quad$ $B_1 = B(d-r)$

$K = D\beta_m^2 t + d$, $\quad$ $s = D\beta_m^2 t + d - h$ $\quad$ $l = D\beta_m^2 t + d + h$ $\quad$ $N = D\beta_m^2 t + d - r$,

When degradation is constant with depth i.e. b=0 Eqn (8) become imperative so for this situation another transformation equation (Eqn.9) was devised to transform the original problem into standard heat flow equation and given as

$$C(z,t) = V(z,t)\exp\left(\frac{vz}{2D} - \left(\frac{v^2}{4D}\right)t\right) + \frac{\gamma_1 e^{-rt}}{r} + (\gamma_0 + R_0)t \qquad \ldots(9)$$

This transformation Eqn.(9) transform the problem into simple heat flow equation under constant degradation rate and gave the final solution of problem as:

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$$C(z,t)=\frac{2}{H}e^{a\,z-d_1\,t}\sum_{m=1}^{\infty}A_2\sin\beta_m z\left(p-B_2\right)\frac{\beta_m}{a^2+\beta_m^2}\left[1-\left(-1\right)^m E\right]$$

$$-\frac{2}{H}e^{a\,z-d_1\,t}\sum_{m=1}^{\infty}A_2\sin\beta_m z\,\frac{\beta_m}{\left(k-a\right)^2+\beta_m^2}\left[1-\left(-1\right)^m Ee^{kH}\right]$$

$$+\left(1-\frac{z}{H}\right)\left(C_1-A_1-B_2e^{-r\,t}\right)e^{az}+E.\frac{z}{H}\left(C_2-A_1-B_2e^{-r\,t}\right)e^{az}$$

$$-\frac{2}{H}e^{a\,z-d_1\,t}\sum_{m=1}^{\infty}\frac{\sin\beta_m z}{\beta_m}\left(g_1-B_2\right)A_2+B_2+A_1\,t$$

$$-\frac{2}{H}e^{a\,z-d_1\,t}\sum_{m=1}^{\infty}\frac{\sin\beta_m z}{\beta_m}A_2\begin{bmatrix}\frac{A_7\left(e^{S_1\,t}-1\right)}{S_1}-\frac{B_3\left(e^{N_1\,t}-1\right)}{N_1}-\frac{A_1\left(e^{K_1\,t}-1\right)}{K_1}\\-\frac{A_8}{K_1{}^2}+\frac{A_8e^{K_1\,t}}{K_1{}^2}-\frac{A_8te^{K_1\,t}}{K_1}\end{bmatrix}$$

$$+\frac{2}{H}e^{a\,z-d_1\,t}\sum_{m=1}^{\infty}\left(-1\right)^m\frac{\sin\beta_m z}{\beta_m}\left(g-B_2\right)E\,A_2$$

$$+\frac{2}{H}e^{a\,z-d_1\,t}\sum_{m=1}^{\infty}\left(-1\right)^m\frac{\sin\beta_m z}{\beta_m}A_2E\begin{bmatrix}\frac{A_9\left(e^{-M\,t}-\right)}{M}-\frac{B_3\left(e^{-N_1}-1\right)}{N_1}-\frac{A_1\left(e^{K_1\,t}-1\right)}{K_1}\\-\frac{A_8}{K_1{}^2}+\frac{A_8e^{K_1\,t}}{K_1{}^2}-\frac{A_8te^{K_1\,t}}{K_1}\end{bmatrix}\qquad\ldots(10)$$

$A_1=\left(\gamma_0+R_0\right)$, $\quad A_7=g_1\left(d_1-k\right)\quad B_3=B_2\left(d_1-r\right)\quad A_8=A_1d_1$

$B_2=\frac{\gamma_1}{r}\quad d_1=\frac{v^2}{4D}\quad S_1=D\beta_m^2+d_1-k$, $\quad K_1=D\beta_m^2+d_1$

$N_1=D\beta_m^2+d_1-r\quad M=D\beta_m^2+d_1+k$

Equation (8) and equation (10) give the complete solution of transport equation (2) under constant and depth dependent degradation rate for combination of constant and exponentially denitrification rate. In further analysis they would be treated as Model 1 and Model 2, respectively.

Comment [DL5]: Where is the objective or are objectives of the research?

Experimental plot: The size of experimental plot was 5 m x 5 m, surrounded by 1 meter buffer zone earlier used by *Behera* (2003), and *Garg et al*. (2005) and lined by galvanized iron sheet as discussed by *Jaynes et al*. (1992). The line of tensiometers and soil-water samplers were put 1.5 away from the side boundary, double ring infiltrometer was kept at the center of the plot while access tubes were installed on the center line of the plot. Depth of both tensiometers and samplers were kept 15, 30, 50, 75, 100 and 150 cm below the ground surface. First and sixth were installed 50 cm away from the

Comment [DL6]: What reasons for these depths?

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boundary and the distance between two were kept 80 cm. access tubes were installed 125 cm from the boundary. Observation wells were installed at two corners diagonally, keeping in mind the general flow direction of water movement. All soil water samplers were connected by a lateral line through HDPP (high density polyvenyle pipe) and connected to vacuum pump which creates suction and pressure in sampler for collection of leachate sample.

Collection of field data: Nitrogen solution of 448 ppm concentration, representing the nitrogen dose of 334 N kg/ ha, was applied in the experimental plot instantaneously to simulate the fertigation. Leachate samples were collected with the help of soil-water sampler and vacuum pump. Collected samplers were brought to laboratory and analyzed for total nitrogen content with the help of *Kjeldhal* unit.

Result and Discussion

Verification of analytical solution with experimental

Physical, chemical, textural and transport parameters, required to validate the developed models, were obtained by standard methods. Computer programmes for model-1 and model-2 were developed in C^{++} language with defining the all input parameters in programme except space and time. Just by giving the value of space and time one can get the concentration of fertilizer at that space and time. Performance of developed models was compared with experimental results and shown in Fig.-1 to Fig.13. First 6 figurers are showing the performance of developed models at 0.15 m, 0.30 m, 0.50 m, 0.75 m, 1.00 m and 1.50 m respectively. At 0.15 m first four days both model-1 and model-2 were over predicting a little more than observed value but form third day onwards both predicted very closed to the observed values which may be clearly seen in Fig.1. Similar performance of models were also observed for other depths except 1.5 m and is clearly depictured in Fig.-2 to Fig.-5 that may be due to preferential flow (funneling, fingering and channeling) of water through the soil or highly disturbed upper soil layer during the installation of soil-water sampler or combination of theses two. Similar performance of models was also depictured in Fig.-7 to Fig.-13 at different day and further validated their performance. Table 1 shows the percent deviation of concentration predicted by developed models and observed values. Deviation is very less except for first two days.

Table-5: Observed and predicted concentration (ppm) by equation 8 and equation 10

Time(days)	Model1	Model 2	Observed	% deviation	% deviation
1	409.46	395.2	375	9.19	5.39
2	470.26	464.23	451	4.27	2.93
3	481.4	477.44	473	1.78	0.94
4	483.73	480.1	478	1.20	0.44
5	484.34	480.76	481	0.69	-0.05
7	484.84	480.83	482	0.59	-0.24
10	485.64	480.73	483	0.55	-0.47

Limiting conditions

Analytical solutions given by Eqns.(8) and (10) under different conditions can be used to obtain the following analytical solutions as special cases: (1) Analytical solutions when

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the nitrification rate is constant by substituting $\gamma_1 = 0$ in above equations. Graphical comparison of developed models with observed value for this condition is shown in Fig- 14 to Fig- 26. (2) Analytical solutions when the nitrification rate is exponentially decreasing by substituting $\gamma_0 = 0$ in above equations. Graphical comparison of developed models with observed value for this condition is shown in Fig- 27 to Fig- 39. (3) Analytical solutions when there is no nitrification by substituting γ_0 *and* $\gamma_1 = 0$ in above equations. Graphical comparison of developed models with observed value for this condition is shown in Fig- 39 to Fig- 52 and (4) Analytical solutions for non-absorbing solutes by substituting R_0 *and* $b = 0$ in above equations. Variations in concentrations under limiting conditions were negligible for model 2 as compared to model 1 in similar situations. Model-2 performed better than Model-1 at each day. Hence it may be concluded that the under local soil condition there is no degradation with depth for nitrogen concentration in shallow groundwater table condition.

Conclusion

Developed models would be successfully used for the prediction of fertilizer movement in irrigated field where water table is high with the accurate knowledge of local transport parameters. Deviation in observed and predicted concentrations was highest at the first day and decreases continuously as time passes this may be due to the highly disturbed top layer caused due to installation of instruments and G.I. sheet. Hence, preferential flow of solute must be minimized before taking the actual observation to avoid such outcome.

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