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2 **Analytical solution of transport equation for exponentially decreasing initial**
3 **concentration in shallow water table condition in irrigated field**
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9
10 **Abstract**

11 Most of the agricultural activities is limited for the depth of 15 – 20 cm and rest soil is remains enact for a
12 long periods which inhibits the microbial activities below this depths and create a initial concentration of
13 nutrients exponentially decreasing with depth. Atmpt has been made to develop analytical models for
14 time dependent nitrification/ denitrification and depth dependent absorption of urea fertilizer in high water
15 table conditions with fertigation. Laplace transformation method was used to solve the unsteady-state
16 advection-dispersion equation. The analytical solutions that can be derived by this method assist
17 understanding of movement of fertilizer in irrigated fields. The developed models were validated with the
18 experimental results. They were closely predicting the fertilizer movement in one-dimension soil medium.
19 The little deviation of result from observed values may be due to change of dispersion coefficient and
20 velocity with moisture content. Here these parameters were assumed as constant throughout the time under
21 consideration. Models developed for constant degradation rate is predicting very close to observe values
22 which shows that the soil under study have no depth dependent degradation. The developed models may be
23 helpful for planning of drain design, nutrient management and assessment of potential hazards to
24 groundwater in agricultural fields by the knowledge of exact transport parameters and boundary conditions
25 universally.
26

27 **Introduction**

28 The top soil up to a depth of 15-20 cm supports the most of the seasonal and shallow
29 deep rooted cereals crops. These types of crops may be grown successfully without
30 tillage in a wide range of soil. Popularity of zero-till machines in Indian agriculture
31 avoids the tillage of soil up to a great extent. Addition of different biofertilizers and soil
32 ameliorators increase the biological activities in upper soil and leave the lower soil enact
33 and creates the situation of exponentially decreasing initial concentration of different soil
34 nutrients in soil mass in high water table areas. Mathematical modelling of processes, in
35 the unsaturated zone, is useful for the agricultural management of cultivated sites, for
36 prediction of the fate of agrochemicals, and for assessment of the potential hazard of
37 shallow groundwater contamination. The difficulty of solving the transport equation in
38 the unsaturated zone relies on its strong nonlinearity. Although significant efforts have
39 been made to overcome the mathematical difficulties, most analytical solutions are
40 derived for one-dimensional vertical transport under various simplifying assumptions.
41 Accounting for the spatial heterogeneity of natural soils renders the transport problem
42 even more complicated.

43 *Bresler and Laufer (1974)* simulated the movement of nitrate in homogeneous soil profile
44 in the presence of $\text{NO}_3\text{-N}$ production (nitrification). *Sexton et al. (1977)* modelled nitrate
45 - nitrogen movement and dissipation in fertilized agricultural lands, but did not include
46 representation of any other fertilizer from nitrogen. *Wagenet et al. (1977)* extended the
47 mathematical analysis of *Cho (1971)* and *Misra et al. (1974)* to describe the transport and
48 transformation of urea, ammonium nitrogen and nitrate nitrogen soil profile as a function
49 of depth and time subject to either steady or pulse feed application of nitrogen, and
50 validated with controlled laboratory experiments.

51 *Davidson et al.* (1978) developed research and development type models on the fate of
52 nitrogen in the root zone by simplifying assumptions of water and solute processes in the
53 field. *Watts and Hanks* (1978), *Tillotson et al* (1980), *Tillotson and Wagenet* (1982)
54 developed a model that simulated most of the major transformations of the nitrate as well
55 as the uptake by the crop, but fell short of fully describing the system in the plant growth
56 and yield response. *Selim and Iskander* (1981) developed the model for calculating
57 pollution from organic wastes and from excessive fertilization. *Tanji et al.* (1982)
58 presented a steady state nitrogen model developed on a mass balance approach, which
59 considered water and nitrogen flow on annual or cropped cycle time basis. *Barraclough*
60 (1989), *Borg et al.* (1990), *Benbi et al.* (1991), *Carbon et al.* (1991) also developed the
61 soil water nitrogen models. *Izadi et al.* (1996) combined functional sub model and
62 analytical solution to the steady state convection dispersion equations to predict the
63 nitrate concentration in the root zone. *Lessoff and Indelman* (2004) investigated the
64 penetration of reactive solute into a soil during a cycle of water infiltration and
65 redistribution by deriving analytical closed form solutions for fluid flux, moisture content
66 and contaminant concentration. *Sander and Braddock* (2005) presented a range of
67 analytical solutions to the combined transient water and solute transport for horizontal
68 flow. *Smedt* (2007) reported an analytical solution and analysis of solute transport in
69 rivers affected by diffusive transfer in the hyporheic zone. *Khakpour and Kambiz* (2008)
70 reported analytical solution of transport phenomena within PEM fuel cell. *Zhan et al*
71 (2009) deduced an analytical solution of two-dimensional solute transport in an aquifer–
72 aquitard system. *Srinivasan and Clement* (2008) reported analytical solutions for
73 sequentially coupled one-dimensional reactive transport problems. *Sadek* (2009)
74 compared various available analytical solution with numerical methods is deduced that
75 analytical solution may be use as a versatile tool for assessment of contaminant transport.
76 *Jozse and Janos* (2009) derived analytical solution of Analytical solution of the coupled
77 2-D turbulent heat and vapor transport equations and the complementary relationship of
78 evaporation . *Guerrero and Skaggs* (2010) presented a general analytical solution for
79 linear, one dimensional advection dispersion equation with distance dependent
80 coefficients. An integrating factor was employed to obtain a transport equation that has a
81 self-adjoint differential operator, and a solution was found using the generalized integral
82 transform technique.

83
84 The mechanisms of solute transport in irrigated field are significantly influenced by
85 attenuation processes such as adsorption and nitrification/denitrification processes. Most
86 of the available analytical solutions are based on linear equilibrium adsorption and first
87 order nitrification and possibly zeroth order production (*van Genuchten and Alves* (1982)
88 for a number of analytical solutions). Here movement of urea fertilizer was analytically
89 solved under depth dependent adsorption factor and combination of constant and
90 exponential nitrification/denitrification rate for exponentially decreasing initial condition.
91 Following assumptions were considered for formulating the boundary value problems:

- 92 1. The soil is unconfined, homogeneous and isotropic overlying an impermeable layer
93 which is having water table depth H meter from soil surface,
- 94 2. The water through deep percolation moves vertically downward until it joins the
95 ground water,
- 96 3. Darcy and Fick's laws hold good,

97 4. Fluid is of constant density and viscosity,
 98 In the present study 1-D Richard's equation in combination with solute transport
 99 equation, which incorporates nitrification and de-nitrification, and depth dependent soil
 100 and water matrix factor was used to characterize the movement of applied fertilizer in
 101 irrigated agriculture having shallow water table conditions.

102 **Governing Equation**

103 Transport equation in unsaturated porous medium is given by:

104

$$105 \quad \frac{\partial}{\partial t}(\theta C + \rho S) = \frac{\partial}{\partial z} \left(\theta D \frac{\partial C}{\partial z} - qC \right) - \alpha \theta C - \rho \beta C \mp \gamma \theta \quad \dots(1)$$

106 where $C = C(z, t)$ is the concentration of chemical in the liquid phase in mg/l, $S =$
 107 $S(z, t)$ is the concentration of chemical in the solid phase in mg/l, $D = D(z, t)$ is the
 108 dispersion coefficient in m^2/day , $\theta = \theta(z, t)$ is the volumetric water content cm^3
 109 $/\text{cm}^3$, $q = q(z, t)$ is the flux of water in m/day, $\rho = \rho(z)$ is the soil bulk density
 110 in gm/cm^3 , $\alpha = \alpha(z)$ is the first-order degradation rate constant in the liquid
 111 phase, $\beta = \beta(z)$ is the first-order degradation rate constant in the solid phase,
 112 $\gamma = \gamma(z)$ is the zero-order production rate constant in the liquid phase.
 113 Here α, β and γ are zero or greater.

114 Considering that soil medium remains intact with time, and introducing mass balance
 115 equation for one dimensional unsteady unsaturated flow condition as given by Chow *et*
 116 *al.*(1988), Eqn. (1) reduces to:

117

$$118 \quad \frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial z^2} - v \frac{\partial C}{\partial z} - RC \mp \gamma \quad \dots(2)$$

119 where $v = q/\theta$ and $R = \left(\alpha + \frac{\rho\beta}{\theta} \right)$, the factor representing the combined effect of liquid
 120 and solid phase degradation rate. Here we assume $R(C) = R_o - b(C_o - C)$ where R_o
 121 represents potential degradation rate at the land surface; b is reduction factor due to
 122 which degradation decreases linearly as the depth from the land surface it increases upto
 123 a specific value; and C_o is initial concentration at the ground surface. For the
 124 development of model, combination of constant and exponentially decreasing
 125 nitrification/de-nitrification rate which may be given as $\gamma(t) = \gamma_0 + \gamma_1 e^{-rt}$, where γ_0 and
 126 γ_1 are constant nitrification/de-nitrification rates, r is decay constant and t represents
 127 time.

128

129 The initial and boundary conditions in mathematical terms, for the solute flow problem in
 130 unsaturated zone under above situation, may be written as:

$$\left. \begin{aligned}
& C(z,0) = C_z(z,0) \quad \text{at } t = 0 \quad \text{for } 0 < z < H \\
131 \quad & C(0,t) = C_1 \quad \text{at } t > 0 \quad \text{for } z = 0 \\
& C(H,t) = C_2 \quad \text{at } t > 0 \quad \text{for } z = H
\end{aligned} \right\} \dots(3)$$

132 where, $C_1 = g_1 e^{-ht}$, $C_2 = g e^{ht}$, g_1 and g are the concentrations at ground surface and H
133 meter below the soil surface before application of fertigation. $C_z(z,0)$ is the distribution
134 of initial concentration in the porous medium. Devising a transform given by Eqn.(4)
135 converted the Eqn. (2) and Eqn.(3) into standard heat flow equation and given by Eqn (5)
136 and Eqn(6), respectively.

$$137 \quad C(z,t) = V(z,t) \exp\left(\frac{vz}{2D} - \left(\frac{v^2}{4D} + b\right)t\right) + \frac{\gamma_1 e^{-rt}}{b-r} + \frac{\gamma_0 + bC_0 - R_0}{b} \dots(4)$$

$$138 \quad \frac{\partial^2 V}{\partial z^2} = \frac{1}{D} \frac{\partial V}{\partial t} \dots(5)$$

$$\left. \begin{aligned}
& V(z,0) = (C_z - A - B)e^{-az} = f(z) \\
140 \quad & V(0,t) = (C_1 - A - B e^{-rt})e^{dt} = f_1(t) \\
& V(H,t) = E(C_2 - A - B e^{-rt})e^{dt} = f_2(t)
\end{aligned} \right\} \dots(6)$$

$$141 \quad \text{where, } A = \frac{\gamma_0 + bC_0 - R_0}{b}, \quad d = \frac{v^2}{4D} + b, \quad a = \frac{v}{2D}, \quad E = e^{-aH} \quad \text{and} \quad B = \frac{\gamma_1}{b-r}$$

142 The general solution of transformed Eqn (5) under initial and boundary condition Eqn(6)
143 is given by *Carslaw* and *Jaeger* (1959) and *Ozisik* (1980) as below:
144

$$\begin{aligned}
145 \quad V(z,t) &= \frac{2}{H} \sum_{m=1}^{\infty} e^{-D\beta_m^2 t} \sin \beta_m z \left[\int_0^H f(z) \sin \beta_m z dz \right] + \left(1 - \frac{z}{H}\right) f_1(t) + \frac{z}{H} f_2(t) \\
146 \quad &- \frac{2}{H} \sum_{m=1}^{\infty} \frac{\sin \beta_m z}{\beta_m} \left[f_1(0) e^{-D\beta_m^2 t} + \int_0^t e^{-D\beta_m^2(t+\tau)} df_1(\tau) \right] \\
147 \quad &+ \frac{2}{H} \sum_{m=1}^{\infty} (-1)^m \frac{\sin \beta_m z}{\beta_m} \left[f_2(0) e^{-D\beta_m^2 t} + \int_0^t e^{-D\beta_m^2(t+\tau)} df_2(\tau) \right] \dots(7)
\end{aligned}$$

148 where, β_m is the root of $\sin \beta_m H = 0$ and τ is a dummy variable. Solution of transport
 149 equation was obtained for exponentially decreasing initial concentration of nitrogen in
 150 soil profile with the help of equation (7) and transformed initial and boundary conditions.
 151 When $C_z = p - e^{kz}$, i.e. exponentially decreasing with depth, then final solution of Eqn.
 152 (7) takes the following form:

$$\begin{aligned}
 153 \quad C(z,t) &= \frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} A_2 \sin \beta_m z \left[(p - A - B) \frac{\beta_m}{a^2 + \beta_m^2} \left[1 - (-1)^m E \right] \right. \\
 154 \quad &- \frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} A_2 \sin \beta_m z \frac{\beta_m}{(k-a)^2 + \beta_m^2} \left[1 - (-1)^m E e^{kH} \right] \\
 155 \quad &+ \left(1 - \frac{z}{H} \right) (C_1 - A - B e^{-rt}) e^{az} + \frac{z}{H} (C_2 - A - B e^{-rt}) e^{a(z-H)} \\
 156 \quad &- \frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} \frac{\sin \beta_m z}{\beta_m} [(g_1 - A - B) A_2] \\
 157 \quad &- \frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} A_2 \frac{\sin \beta_m z}{\beta_m} \left[\frac{A_6 A_2}{s} (e^{-st} - 1) - \frac{A_4 A_2}{K} (e^{-Kt} - 1) - \frac{A_2 B_1}{N} (e^{-Nt} - 1) \right] \\
 158 \quad &+ \frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} (-1)^m A_2 \frac{\sin \beta_m z}{\beta_m} [(g - A - B) E] + A + B e^{-rt} \\
 159 \quad &+ \frac{2}{H} e^{az-dt} \sum_{m=1}^{\infty} (-1)^m A_2 E \frac{\sin \beta_m z}{\beta_m} \left[\frac{A_3}{l} (e^{-lt} - 1) - \frac{B_1}{N} (e^{-Nt} - 1) - \frac{A_4}{K} (e^{-Kt} - 1) \right] \quad \dots(8)
 \end{aligned}$$

$$160 \quad A_2 = e^{-D\beta_m^2 t} \quad A_3 = (h+d)g \quad A_4 = d \quad A_5 = (C_0 - A - B)$$

$$161 \quad A_5 = (C_0 - A - B) \quad A_6 = g_1(d-h), \quad B_1 = B(d-r)$$

$$162 \quad K = D\beta_m^2 t + d, \quad s = D\beta_m^2 t + d - h \quad l = D\beta_m^2 t + d + h \quad N = D\beta_m^2 t + d - r,$$

163 When degradation is constant with depth i.e. $b=0$ Eqn (8) become imperative so for this
 164 situation another transformation equation (Eqn.9) was devised to transform the original
 165 problem into standard heat flow equation and given as

$$166 \quad C(z,t) = V(z,t) \exp \left(\frac{vz}{2D} - \left(\frac{v^2}{4D} \right) t \right) + \frac{\gamma_1 e^{-rt}}{r} + (\gamma_0 + R_0) t \quad \dots(9)$$

167 This transformation Eqn.(9) transform the problem into simple heat flow equation under
 168 constant degradation rate and gave the final solution of problem as:

$$\begin{aligned}
169 \quad C(z,t) &= \frac{2}{H} e^{az-d_1t} \sum_{m=1}^{\infty} A_2 \sin \beta_m z (p - B_2) \frac{\beta_m}{a^2 + \beta_m^2} [1 - (-1)^m E] \\
170 \quad &- \frac{2}{H} e^{az-d_1t} \sum_{m=1}^{\infty} A_2 \sin \beta_m z \frac{\beta_m}{(k-a)^2 + \beta_m^2} [1 - (-1)^m E e^{kH}] \\
171 \quad &+ \left(1 - \frac{z}{H}\right) (C_1 - A_1 - B_2 e^{-rt}) e^{az} + E \cdot \frac{z}{H} (C_2 - A_1 - B_2 e^{-rt}) e^{az} \\
172 \quad &- \frac{2}{H} e^{az-d_1t} \sum_{m=1}^{\infty} \frac{\sin \beta_m z}{\beta_m} (g_1 - B_2) A_2 + B_2 + A_1 t \\
173 \quad &- \frac{2}{H} e^{az-d_1t} \sum_{m=1}^{\infty} \frac{\sin \beta_m z}{\beta_m} A_2 \left[\frac{A_7 (e^{S_1 t} - 1)}{S_1} - \frac{B_3 (e^{N_1 t} - 1)}{N_1} - \frac{A_1 (e^{K_1 t} - 1)}{K_1} \right. \\
174 \quad &\quad \left. - \frac{A_8}{K_1^2} + \frac{A_8 e^{K_1 t}}{K_1^2} - \frac{A_8 t e^{K_1 t}}{K_1} \right] \\
175 \quad &+ \frac{2}{H} e^{az-d_1t} \sum_{m=1}^{\infty} (-1)^m \frac{\sin \beta_m z}{\beta_m} (g - B_2) E A_2 \\
&+ \frac{2}{H} e^{az-d_1t} \sum_{m=1}^{\infty} (-1)^m \frac{\sin \beta_m z}{\beta_m} A_2 E \left[\frac{A_9 (e^{-M t} - 1)}{M} - \frac{B_3 (e^{-N_1 t} - 1)}{N_1} - \frac{A_1 (e^{K_1 t} - 1)}{K_1} \right. \\
&\quad \left. - \frac{A_8}{K_1^2} + \frac{A_8 e^{K_1 t}}{K_1^2} - \frac{A_8 t e^{K_1 t}}{K_1} \right] \quad \dots(10)
\end{aligned}$$

$$176 \quad A_1 = (\gamma_0 + R_0), \quad A_7 = g_1(d_1 - k) \quad B_3 = B_2(d_1 - r) \quad A_8 = A_1 d_1$$

$$177 \quad B_2 = \frac{\gamma_1}{r} \quad d_1 = \frac{v^2}{4D} \quad S_1 = D\beta_m^2 + d_1 - k, \quad K_1 = D\beta_m^2 + d_1$$

$$178 \quad N_1 = D\beta_m^2 + d_1 - r \quad M = D\beta_m^2 + d_1 + k$$

179 Equation (8) and equation (10) give the complete solution of transport equation (2) under
180 constant and depth dependent degradation rate for combination of constant and
181 exponentially denitrification rate. In further analysis they would be treated as Model 1
182 and Model 2, respectively.

183 **Experimental plot:** The size of experimental plot was 5 m x 5 m, surrounded by 1 meter
184 buffer zone earlier used by *Behera* (2003), and *Garg et al.* (2005) and lined by
185 galvanized iron sheet as discussed by *Jaynes et al.* (1992). The line of tensiometers and
186 soil-water samplers were put 1.5 away from the side boundary, double ring infiltrometer
187 was kept at the center of the plot while access tubes were installed on the center line of
188 the plot. Depth of both tensiometers and samplers were kept 15, 30, 50, 75, 100 and 150
189 cm below the ground surface. First and sixth were installed 50 cm away from the

190 boundary and the distance between two were kept 80 cm. access tubes were installed 125
 191 cm from the boundary. Observation wells were installed at two corners diagonally,
 192 keeping in mind the general flow direction of water movement. All soil water samplers
 193 were connected by a lateral line through HDPP (high density polyvenyle pipe) and
 194 connected to vacuum pump which creates suction and pressure in sampler for collection
 195 of leachate sample.

196
 197 **Collection of field data:** Nitrogen solution of 448 ppm concentration, representing the
 198 nitrogen dose of 334 N kg/ ha, was applied in the experimental plot instantaneously to
 199 simulate the fertigation. Leachate samples were collected with the help of soil-water
 200 sampler and vacuum pump. Collected samplers were brought to laboratory and analyzed
 201 for total nitrogen content with the help of *Kjeldhal* unit.

202 **Result and Discussion**

203 **Verification of analytical solution with experimental**

204 Physical, chemical, textural and transport parameters, required to validate the developed
 205 models, were obtained by standard methods. Computer programmes for model-1 and
 206 model-2 were developed in C++ language with defining the all input parameters in
 207 programme except space and time. Just by giving the value of space and time one can get
 208 the concentration of fertilizer at that space and time. Performance of developed models
 209 was compared with experimental results and shown in Fig.-1 to Fig.13. First 6 figures
 210 are showing the performance of developed models at 0.15 m, 0.30 m, 0.50 m, 0.75 m,
 211 1.00 m and 1.50 m respectively. At 0.15 m first four days both model-1 and model-2
 212 were over predicting a little more than observed value but from third day onwards both
 213 predicted very closed to the observed values which may be clearly seen in Fig.1. Similar
 214 performance of models were also observed for other depths except 1.5 m and is clearly
 215 depicted in Fig.-2 to Fig.-5 that may be due to preferential flow (funneling, fingering
 216 and channeling) of water through the soil or highly disturbed upper soil layer during the
 217 installation of soil-water sampler or combination of theses two. Similar performance of
 218 models was also depicted in Fig.-7 to Fig.-13 at different day and further validated their
 219 performance. Table 1 shows the percent deviation of concentration predicted by
 220 developed models and observed values. Deviation is very less except for first two days.

221 Table-5: Observed and predicted concentration (ppm) by equation 8 and equation 10

Time(days)	Model1	Model 2	Observed	% deviation	% deviation
1	409.46	395.2	375	9.19	5.39
2	470.26	464.23	451	4.27	2.93
3	481.4	477.44	473	1.78	0.94
4	483.73	480.1	478	1.20	0.44
5	484.34	480.76	481	0.69	-0.05
7	484.84	480.83	482	0.59	-0.24
10	485.64	480.73	483	0.55	-0.47

222

223 **Limiting conditions**

224 Analytical solutions given by Eqns.(8) and (10) under different conditions can be used to
 225 obtain the following analytical solutions as special cases: (1) Analytical solutions when

226 the nitrification rate is constant by substituting $\gamma_1 = 0$ in above equations. Graphical
227 comparison of developed models with observed value for this condition is shown in Fig-
228 14 to Fig- 26. (2) Analytical solutions when the nitrification rate is exponentially
229 decreasing by substituting $\gamma_0 = 0$ in above equations. Graphical comparison of developed
230 models with observed value for this condition is shown in Fig- 27 to Fig- 39. (3)
231 Analytical solutions when there is no nitrification by substituting γ_0 and $\gamma_1 = 0$ in
232 above equations. Graphical comparison of developed models with observed value for this
233 condition is shown in Fig- 39 to Fig- 52 and (4) Analytical solutions for non-absorbing
234 solutes by substituting R_0 and $b = 0$ in above equations. Variations in concentrations
235 under limiting conditions were negligible for model 2 as compared to model 1 in similar
236 situations. Model-2 performed better than Model-1 at each day. Hence it may be
237 concluded that the under local soil condition there is no degradation with depth for
238 nitrogen concentration in shallow groundwater table condition.
239

240 Conclusion

241 Developed models would be successfully used for the prediction of fertilizer movement
242 in irrigated field where water table is high with the accurate knowledge of local transport
243 parameters. Deviation in observed and predicted concentrations was highest at the first
244 day and decreases continuously as time passes this may be due to the highly disturbed top
245 layer caused due to installation of instruments and G.I. sheet. Hence, preferential flow of
246 solute must be minimized before taking the actual observation to avoid such outcome.

247

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