__Original Research Article__

The Generalized Inverse Gaussian Frailty with Application in Life Annuity Business

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ABSTRACT

Aims: As shown in literature, several authors have adopted various individual frailty mixing distributions as a way of dealing with possible heterogeneity due to unobserved covariates in a group of insureds. This research contribution is to generalize the frailty mixing distribution to nest other classes of frailty distributions not in literature and apply the proposed distributions in valuation of life annuity business.

Methodology: A simulation study is done to assess the performance of the aforementioned models. The baseline parameters is estimated using Bayesian Inference and a better model is suggested for valuation of life annuity business.

Results:

As a result of generalizing the frailty some new classes of frailty distributions are constructed such as; the Reciprocal Inverse Gaussian Frailty, the Inverse Gamma Frailty, the Harmonic Frailty and the Positive Hyperbolic Frailty.

From a simulation study the proposed models shows that ignoring frailty leads to an underestimation of future residual lifetime as expected thereby underestimating the insurer's liability in the context of life annuity business.

The Reciprocal Inverse Gaussian frailty model closely represents the Association of Kenya Insurers-2010 graduated rates between ages 55-80 beyond which there is longevity risk.

Conclusion:

The proposed special cases of the Generalized Inverse Gaussian frailty show an increase in the insurers liability as expected when unobserved heterogeneity is accounted for. The Reciprocal Inverse Gaussian Frailty model as proposed in estimating future liability by directly adjusting the Association of Kenya Insurers mortality rates shows an increase in longevity risk more at extreme ages (beyond 80). The Generalized Inverse Gaussian frailties should be considered for multivariate cases where the insureds are clustered in groups.

Keywords: Frailty Model, Generalized Inverse Gaussian Distribution, Reciprocal Inverse Gaussian Distribution, Harmonic Distribution, Positive Hyperbolic Distribution , Bayesian Inference, Life Annuity Insurance.

1. INTRODUCTION

Frailty modeling is based on mixture distributions where the population hazard is considered a mixture of measurable (e.g. health status) and unmeasured (e.g. congenital personal characteristics) risk factors affecting mortality. The frailty model is an extension of the Cox PH model. It is a random effects model which has a multiplicative effect on the hazard rate that adds additional risks based on each individual's information. The term "frailty" was introduced by [1] in a seminal paper on individual survival models. They discuss the impact of heterogeneity in individual mortality. Their findings showed that standard life-table methods overestimates current life expectancy and potential gains in life expectancy from health and safety interventions, while underestimating rates of individual aging, past progress in reducing mortality.

Standard life tables assume that the population under study is homogeneous. This means that all individuals in that study are subject under the same risk at a given age. Basic observation of medical statistics shows that individuals differ greatly.

Frailty models have been adopted by several authors in insurance, for instance; [2] applies frailty to quantify the extent of heterogeneity in Australian population mortality on life annuity rates and pension costs the results confirm significant heterogeneity exists. [3] have used frailty model to quantify the impact of heterogeneity due to underwriting factors and frailty on annuity values the results showed that heterogeneity remains after underwriting and that frailty significantly impacts the fair value of both standard and underwritten annuities.

[4] suggests adopting a frailty model for risk classification for life annuity portfolios. In particular, they identify risk groups within a population by assigning specific ranges of values to the frailty within each group. [5] applies frailty modelling to analyze the impact of frailty, in its various interpretations, on the results of cash flows, profits, etc of life insurance and life annuity portfolios and related risk profiles.

[6] bootstrap data on Canadian pensioners' mortality to study the characteristics of its implied heterogeneity they find strong support for the Gamma frailty model. [7] utilizes the frailty model to reflect mortality heterogeneity in optimal risk classification for substandard annuities. Their findings indicate that extended frailty risk classification can enhance the insurer's profitability.

[8] applies the additive Gamma frailty models to competing risks in related individuals.

2. METHODOLOGY

The Multiplicative approach.

$$h(t \mid u) = u h_0(t)$$

$h_0(t)$ is the 'standard hazard function' corresponding to a 'standard individual', conventionally those with frailty $u = 1$ .The non-negative quantity u encompasses all other factors affecting mortality other than age which acts in a multiplicative manner. Individuals with $u > 1$ experience a force of mortality that is proportionally higher at all ages. Individuals with $u < 1$ experience proportionally lower mortality rates.

Frailty models without observed covariates.

This model is used when only survival data is available for the analysis, or when additional information is of no interest. I.e $h(t \mid u) = uh_0(t)$

This model is non-identifiable from survival data, since different combinations of $h_0(t)$ and frailty distributions may produce the same marginal hazard rate $h(t)$. The model becomes identifiable when the parametric structure of $h_0(t)$ is fixed and u is assumed to belong to some parametric distribution family.

Univariate Frailty Models

This model accounts for heterogeneity due to unobserved risk factors for independent life times in a proportional hazard model. The variability can be split into a part that depends on observable risk factors, and is therefore theoretically predictable, and a part that is theoretically unpredictable, even when all relevant information is known. This model has been used by several authors [9] and [10] to show that these two sources of variability can explain some unexpected results.

2.1 Model Construction

Let T be the future life-time random variable with a continuous distribution. A non-negative random variable u is called "frailty" if the conditional hazard function given $U = u$ is given by; $h(t \mid u) = uh_0(t); t > 0$ where $h_0(t)$ is called the baseline age-specific hazard function for a "standard" individual. The "population" hazard corresponding to a randomly selected individual that is actually observed is given by;

$E[h(t \mid u)] = \int uh_0(t) f(u)du = \bar{u} * h_0(t)$ increases less rapidly than for individuals. This is because the population becomes populated by more and more robust individuals as the frail members fail. The conditional survival function is given by; $s(t \mid u) = e^{-\int_0^t h(x|u)dx} = e^{-\int_0^t uh_0(x)dx}$

$s(t \mid u) = e^{-uH_0(t)}; t > 0$ where $H_0(t) = \int_0^t h_0(x)dx$ is the cumulative baseline hazard.

Comparing the conditional and unconditional survival functions yields $s(t\,|\,u) = \{s(t)\}^{u}$

This shows that an individual with frailty of level of 2 is twice as likely to die compared to a "standard" individual.

Since the frailty "u" is unobserved and considered random it is integrated out and thus the population is considered a mixture over "u". The univariate marginal survival function is;

$$s(t) = \int_{0}^{\infty} s(t\,|\,u)\, f(u)du = E[s(t\,|\,u)] = E[e^{-uH_0(t)}] = L_u(H_0(t)); t > 0 \qquad (1)$$

The contribution is to construct a generalized frailty mixing distribution to nest other individual classes of frailty distributions.

2.1.1 The Generalized Inverse Gaussian Frailty

The Generalized Inverse Gaussian (GIG) distribution can be constructed under various parametizations. For instance, considering The [11] parametization $\omega = \sqrt{\varphi\theta}$;

$$K_v(\omega) = \frac{1}{2}\int_{0}^{\infty} z^{v-1} e^{-\frac{\omega}{2}(z+\frac{1}{z})} dz$$

where $K_v(\omega)$ is a Bessel function of the third kind with order w and index v.

Using the transformation; $$z = \sqrt{\frac{\varphi}{\theta}}x, dz = \sqrt{\frac{\varphi}{\theta}}dx$$

$$K_v(\sqrt{\varphi\theta}) = \frac{1}{2}\int_{0}^{\infty} (\sqrt{\frac{\varphi}{\theta}})^{v} x^{v-1} e^{-\frac{\sqrt{\varphi\theta}}{2}(x\sqrt{\frac{\varphi}{\theta}}+\frac{1}{x\sqrt{\frac{\varphi}{\theta}}})} dx$$

$$1 = \frac{\int_{0}^{\infty} (\sqrt{\frac{\varphi}{\theta}})^{v} x^{v-1} e^{-\frac{1}{2}(\varphi x+\frac{\theta}{x})} dx}{2K_v(\sqrt{\varphi\theta})}$$

$$f(x)=\frac{(\sqrt{\frac{\varphi}{\theta}})^{v}x^{v-1}e^{-\frac{1}{2}(\varphi x+\frac{\theta}{x})}}{2K_{v}(\sqrt{\varphi\theta})};x>0,\varphi>0,\theta>0,-\infty<v<\infty \qquad (2)$$

Is the probability density function of a GIG(v,φ,θ)

The Laplace transform is

$$L_{u}(s)=E[e^{-su}]=\frac{\int_{0}^{\infty}u^{v-1}(\sqrt{\frac{\varphi}{\theta}})^{v}e^{-\frac{1}{2}[\varphi u+2su+\frac{\theta}{u}]}du}{2K_{v}(\sqrt{\varphi\theta})}$$

$$L_{u}(s)=\frac{(\sqrt{\frac{\varphi}{\theta}})^{v}2K_{v}(\sqrt{(\varphi+2s)\theta}}{(\sqrt{\frac{\varphi+2s}{\theta}})^{v}2K_{v}(\sqrt{\varphi\theta})}=\frac{(\sqrt{\frac{\varphi}{\varphi+2s}})^{v}K_{v}(\sqrt{(\varphi+2s)\theta})}{K_{v}(\sqrt{\varphi\theta})} \qquad (3)$$

The marginal survival function at time $t>0$

$S(t)=L_{U}(H_{0}(t))$

$$S(t)=\frac{(\sqrt{\frac{\varphi}{\varphi+2(H_{0}(t))}})^{v}K_{v}(\sqrt{(\varphi+2H_{0}(t))\theta})}{K_{v}(\sqrt{\varphi\theta})}$$

SPECIAL CASES

Case 1: Inverse Gaussian Distribution (IG):

Let $v=-\frac{1}{2}$ in equation (3)

$$L_{U}(s)=\frac{(\sqrt{\frac{\varphi}{\varphi+2s}})^{-\frac{1}{2}}K_{-\frac{1}{2}}(\sqrt{(\varphi+2s)\theta})}{K_{-\frac{1}{2}}(\sqrt{\varphi\theta})}$$

$$L_U(s)=\frac{\left(\sqrt{\frac{\varphi}{\varphi+2s}}\right)^{-\frac{1}{2}}\sqrt{\frac{\pi}{2(\sqrt{(\varphi+2s)\theta})}}e^{-(\sqrt{(\varphi+2s)\theta})}}{\sqrt{\frac{\pi}{2\sqrt{\varphi\theta}}}e^{-\sqrt{\varphi\theta}}}$$

$$L_U(s)=e^{\sqrt{\varphi\theta}-\sqrt{(\varphi+2s)\theta}} \tag{4}$$

Substituting $\varphi=\frac{1}{\beta^2}, \theta=\mu^2$ we get

$$L_U(s)=\exp\{-\frac{\mu}{\beta}[(1+2\beta^2 s)^{1/2}-1]\}$$

Mean = $-L'_u(0) = \mu\beta$

Variance = $L''_u(0)$ - $(L'_u(0))^2$ = $\mu\beta^3$

For identifiability reasons the mean is normalized to one. i.e. $\mu\beta=1; \mu=\frac{1}{\beta}$ thus the

variance $\delta^2=\beta^2$

The Laplace transform is therefore

$$L_U(s)=\exp[\frac{1-(1+2s\delta^2)^{1/2}}{\delta^2}]$$

The marginal survival function at time $t>0$

$$S(t)=L_U(H_0(t))$$

$$S(t)=\exp[\frac{1-(1+2(H_0(t))\delta^2)^{1/2}}{\delta^2}]$$

Case 2:Reciprocal Inverse Gaussian Distribution (RIG)

Let: $v = \frac{1}{2}$ in equation (3)

$$L_U(s) = \frac{(\sqrt{\frac{\varphi}{\varphi+2s}})^{\frac{1}{2}} K_{\frac{1}{2}}(\sqrt{(\varphi+2s)\theta})}{K_{\frac{1}{2}}(\sqrt{\varphi\theta})}$$

$$L_U(s) = \frac{(\sqrt{\frac{\varphi}{\varphi+2s}})^{\frac{1}{2}} \sqrt{\frac{\pi}{2(\sqrt{(\varphi+2s)\theta})}} e^{-(\sqrt{(\varphi+2s)\theta})}}{\sqrt{\frac{\pi}{2\sqrt{\varphi\theta}}} e^{-\sqrt{\varphi\theta}}}$$

$$L_U(s) = (\frac{\varphi}{2s+\varphi})^{\frac{1}{2}} e^{\sqrt{\varphi\theta}-\sqrt{(\varphi+2s)\theta}}$$

$$L_U(s) = (1+\frac{2s}{\varphi})^{-\frac{1}{2}} e^{\sqrt{\varphi\mu}\{1-\sqrt{(1+\frac{2s}{\varphi})}\}}$$

Substituting $\varphi = \frac{1}{\beta^2}, \theta = \mu^2$ we get

$$L_U(s) = (1+2s\beta^2)^{-\frac{1}{2}} e^{\frac{\mu}{\beta}\{1-\sqrt{1+2s\beta^2}\}}$$

For identifiability the mean is normalized to one i.e.

$$E[U] = -L_U^{'}(s) = \{\beta^2(1+2s\beta^2)^{-\frac{3}{2}} + \mu\beta(1+2s\beta^2)^{-1}\}e^{\frac{\mu}{\beta}\{1-\sqrt{1+2s\beta^2}\}}$$

$$E[U] = -L_U^{'}(0) = \beta^2 + \mu\beta$$

$$\beta^2 + \mu\beta = 1; \mu = \frac{1-\beta^2}{\beta}$$

The Laplace becomes;

$$L_U(s) = (1+2s\beta^2)^{-\frac{1}{2}} e^{\frac{1-\beta^2}{\beta^2}\{1-\sqrt{1+2s\beta^2}\}}$$

The marginal survival function at time $t > 0$

$$S(t) = L_U(H_0(t))$$

$$S(t) = (1+2\beta^2 H_0(t))^{-\frac{1}{2}} e^{\frac{1-\beta^2}{\beta^2}\{1-\sqrt{1+2\beta^2 H_0(t)}\}}$$

Case 3: Gamma Distribution:

Let $\theta = 0, v > 0, \varphi > 0$ in equation (3)

$$L_U(s) = \frac{(\sqrt{\frac{\varphi}{\varphi+2s}})^{v} K_v(\sqrt{(\varphi+2s)*0})}{K_v(\sqrt{\varphi*0})} = (\frac{\varphi}{\varphi+2s})^{\frac{v}{2}}$$

Substituting $\varphi = b, \frac{v}{2} = p$ we get

$L_U(s) = (1+\frac{s}{b})^{-p}$

The mean and variance frailty

$$E[U] = -L_U{}'(0) = p*(1+\frac{s}{b})^{-p-1}*\frac{1}{b} = \frac{p}{b}$$

$$Var(z) = (L''(0) - (L'(0))^2 = -p(-p-1)*(1+\frac{s}{b})^{-p-2}*(\frac{1}{b})^2 - (\frac{p}{b})^2 = \frac{p}{b^2}$$

For purposes of identifiability assume the distribution of U has mean normalized to one

($p = b$) and variance $\sigma^2 = \frac{1}{b}$

The Laplace becomes

$L(s) = (1+s\delta^2)^{-1/\delta^2}$

The marginal survival function at time $t > 0$

$S(t) = L_U(H_0(t))$

$S(t) = \{1+\delta^2 H_0(t)\}^{-1/\delta^2}$

Case 4: The Levy Distribution:

This is a special case of the Inverse Gaussian distribution.

Let $\theta > 0, v = -\frac{1}{2}, \varphi = 0$ in equation (4)

$$L_U(s) = e^{\sqrt{\varphi\theta} - \sqrt{(\varphi+2s)\theta}}$$

when $\varphi = 0$ $L_U(s) = e^{-\sqrt{2s\theta}}$

For purposes of identifiability assume the distribution of U has mean normalized to 1. e.g. $E[U] = \theta = 1$

The Laplace becomes

$$L_U(s) = e^{-\sqrt{2s}}$$

The marginal survival function at time $t > 0$

$$S(t) = L_U(H_0(t))$$

$$S(t) = e^{-\sqrt{2H_0(t)}}$$

Case 5: The Harmonic Distribution

Let $\theta = an, v = 0, \varphi = \frac{a}{n}$ in equation (3)

The Laplace becomes;

$$L_U(s) = \frac{K_0(\sqrt{2ans + a^2})}{K_0(a)}$$

For purposes of identifiability assume the distribution of U has mean normalized to 1.

$$E[U] = \frac{nK_1(1/n)}{K_0(a)} = 1$$

The Laplace becomes

$$L_U(s) = \frac{K_0(\sqrt{2ans + a^2})}{nK_1(1/n)}$$

The marginal survival function at time $t > 0$

$$S(t) = L_Z(H_0(t))$$

$$S(t)=\frac{K_0(\sqrt{2anH_0(t)+a^2})}{nK_1(1/n)}$$

Case 6: The Positive Hyperbolic Distribution

Let $\theta>0, v=1, \varphi>0$ in equation (3)

The Laplace becomes;

$$L_U(s)=\frac{(\sqrt{\frac{\varphi}{\varphi+2s}})K_1(\sqrt{(\varphi+2s)\theta})}{K_1(\sqrt{\varphi\theta})}$$

For purposes of identifiability assume the distribution of U has mean normalized to 1. e.g.

$$E[U]=(\sqrt{\frac{\theta}{\varphi}})\frac{K_2(\sqrt{\varphi\theta})}{K_1(\sqrt{\varphi\theta})}=1$$

The Laplace becomes

$$L_U(s)=\frac{(\sqrt{\frac{\varphi^2}{(\varphi+2s)\theta}})K_1(\sqrt{(\varphi+2s)\theta})}{K_2(\sqrt{\varphi\theta})}$$

The marginal survival function at time $t>0$

$$S(t)=L_U(H_0(t))$$

$$S(t)=\frac{(\sqrt{\frac{\varphi^2}{(\varphi+2H(t))\theta}})K_1(\sqrt{(\varphi+2H(t))\theta})}{K_2(\sqrt{\varphi\theta})}$$

2.1.2 Baseline Hazard Distribution

The baseline hazard used in this study is the Exponential Distribution.

The Exponential Baseline Distribution.

The density function, survival function and hazard function are;

$$f(t)=\lambda e^{-\lambda t};t,\lambda>0$$
$$s(t)=\int_t^\infty \lambda e^{-\lambda u}du=e^{-\lambda t}$$
$$h(t)=\frac{f(t)}{s(t)}=\lambda$$
$$H(t)=\int_0^t h(u)du=\int_0^t \lambda du=\lambda t$$

3. RESULTS AND DISCUSSION

3.1 Simulation Study

A simulation study is done to check the performance of the proposed models. A comparison is done on the survival function in the presence of different levels of frailty (heterogeneity) and without frailty (homogeneity). The baseline parameters are estimated via Bayesian analysis using Gibbs Sampler. RCODE is shown in APPENDIX

CASE1

Gamma-Exponential Model

The Survival function without frailty: $S(t)=e^{-\lambda t}$

The Survival function with frailty: $S(t)=(1+\delta^2\lambda t)^{-1/\delta^2}$

λ=0.05 (failure rate), δ2=5,30,60 (frailty levels) t=55:110 (age)

CASE2

Inverse Gaussian-Exponential Model

The Survival function without frailty: $S(t)=e^{-\lambda t}$

The Survival function with frailty: $S(t)=\exp\{\frac{1-(1+2\delta^2\lambda t)\hat{}0.5}{\delta^2})$

λ=0.05 (failure rate), δ2=5,30,60 (frailty levels) t=55:110 (age)

CASE3

Reciprocal Inverse Gaussian-Exponential Model

The Survival function without frailty: $S(t)=e^{-\lambda t}$

The Survival function with frailty:

$$S(t)=(1+2\beta^{2}\lambda t)^{\wedge}(-0.5)\exp\{\frac{1-\beta^{2}}{\beta^{2}}*(1-(1+2\beta^{2}\lambda t)^{\wedge}0.5)\}$$

λ=0.05 (failure rate), β=0.1, t=55:110 (age)

CASE4

Levy-Exponential Model

The Survival function without frailty: $S(t)=e^{-\lambda t}$

The Survival function with frailty: $S(t)=e^{-(2\lambda t)^{0.5}}$

CASE5

Harmonic-Exponential Model

The Survival function without frailty: $S(t)=e^{-\lambda t}$

The Survival function with frailty:

$$S(t)=\frac{K_0(\sqrt{2an\lambda t+a^2})}{nK_1(1/n)}$$

a=0.05, n=2.5 (shape and scale parameters)

CASE6

Positive Hyperbolic-Exponential Model

The Survival function without frailty: $S(t)=e^{-\lambda t}$

The Survival function with frailty:

$$S(t)=\frac{(\sqrt{\frac{\varphi^{2}}{(\varphi+2\lambda t)\theta}})K_1(\sqrt{(\varphi+2\lambda t)\theta})}{K_2(\sqrt{\varphi\theta})}$$

theta=10, varphi=0.03(shape and scale parameters)

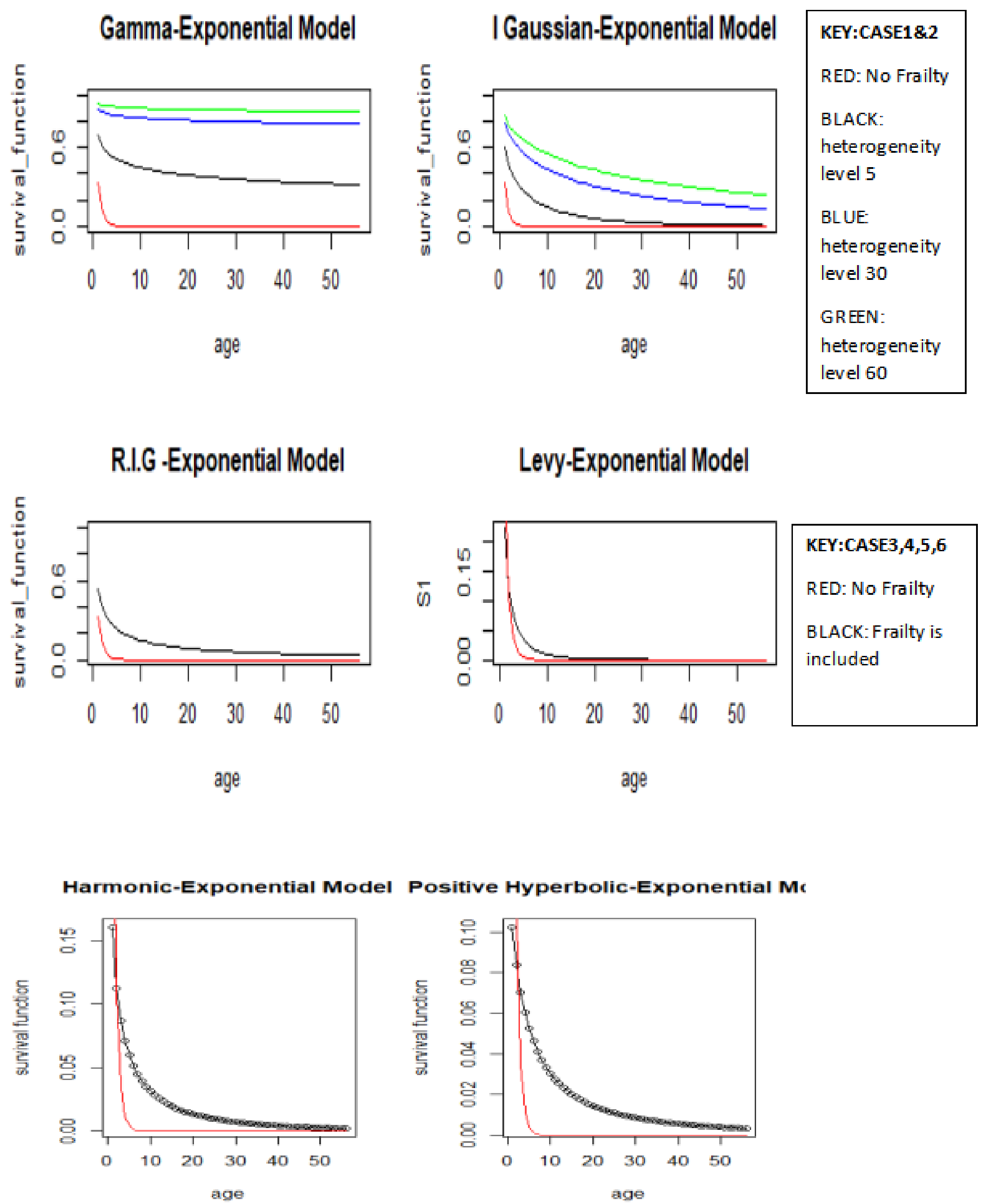


Fig. 1. Graphical representation of exponential model

Discussion

From the simulation study above the proposed models shows that ignoring frailty can lead to an underestimation of future residual lifetime as expected thereby underestimating the insurer's liability in the context of life annuity business.

Parameter Estimation

Model parameters are fixed quantitative values that characterize the model believed to reflect the real world. They have to be estimated either by statistical inference from observations or by expert opinion. Since the baseline model assumes the population to be homogeneous, the Association of Kenya Insurer's (AKI-2010) graduated rates will be considered as the baseline.

3.2 Applications in Insurance Industry

The pricing of long term insurance, annuity and pension products is largely influenced by the choice of the mortality projection model. Frailty models are used in life insurance to represent heterogeneity in a population due to non-observed risk factors. Heterogeneity due to observable risk factors is addressed at policy issue during the underwriting process to ensure that each contract is assigned premium consistent with the insured risk. Neglecting such factors or use of age and sex as the only rating factors (see Joelle F ,2015**Error! Bookmark not defined.**) may lead to mispricing of insurance products.

The aims of this exercise are:

- The first aim is to show that when heterogeneity is disregarded the expected residual lifetime is underestimated thus leading to an underestimation of the insurer's liability.
- Secondly, is to show the relevance of the proposed Reciprocal Inverse Gaussian Frailty mixture to reflect the insurer's mortality rating.

Assumption:

- The force of mortality μ is assumed piece-wise constant, taking a common value across each whole year of age $[x, x+1)$ similar assumption found in [12]

Data Analysis and Results

Consider two hypothetical insurers i.e. insurer X and Y

Insurer X assumes the population to be heterogeneous with respect to underwriting factors and applies the Life-tables from the Association of Kenya Insurers (AKI 2010) graduated rates.

Insurer Y assumes the population to be heterogeneous with respect to both observable and unobserved risk factors and applies frailty modeling to modify the AKI q-rates.

The Reciprocal Inverse Gaussian-Exponential Model

$$S(t) = (1+2\beta^2 H_0(t))\wedge(-0.5)\exp\{\frac{1-\beta^2}{\beta^2} * (1-(1+2\beta^2 H_0(t))\wedge 0.5)\}$$

Where H0(t)~AKI 2010 q-rates

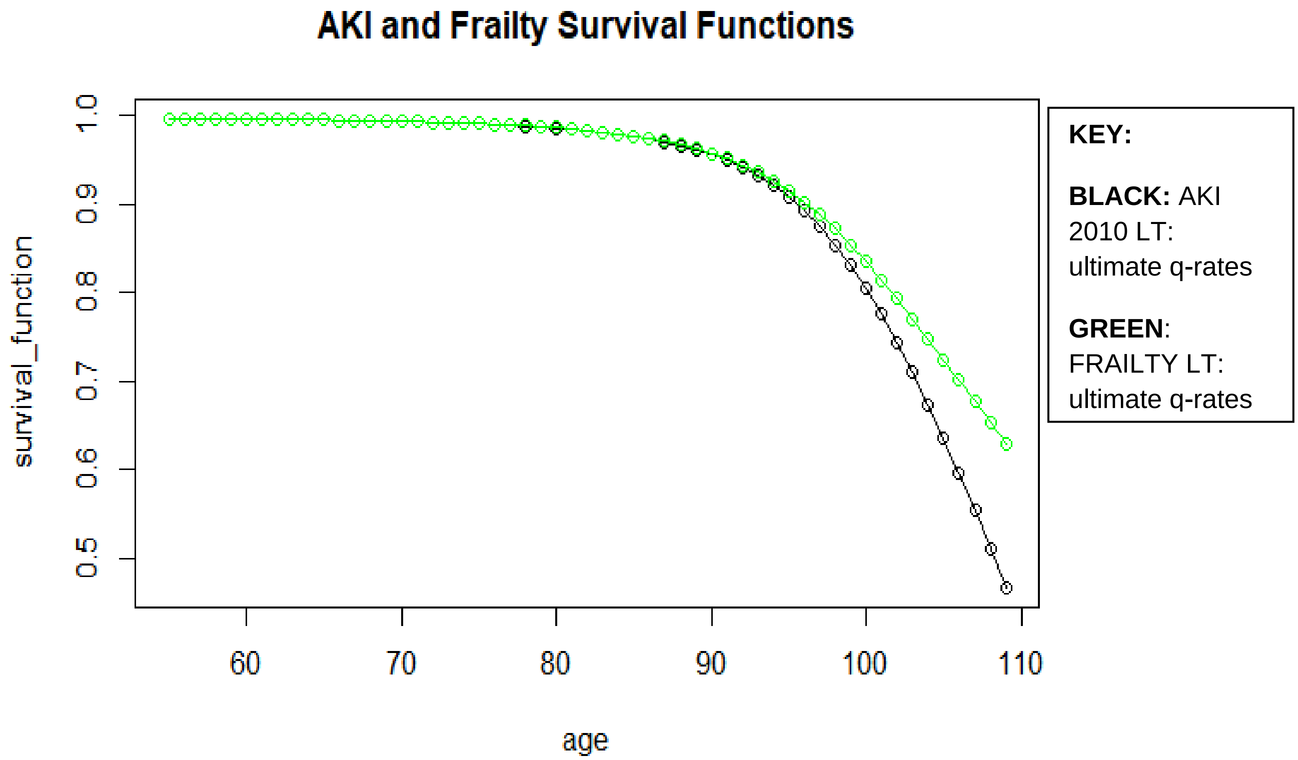


Fig. 2. AKI and Frailty survival functions

Discussion

1. Ignoring heterogeneity due to other factors affecting mortality other then age and sex could lead to underestimation of life expectancy.

2. Ignoring heterogeneity due to unobserved risk factors leads to underestimation of longevity risk thereby underestimating the expected liability.

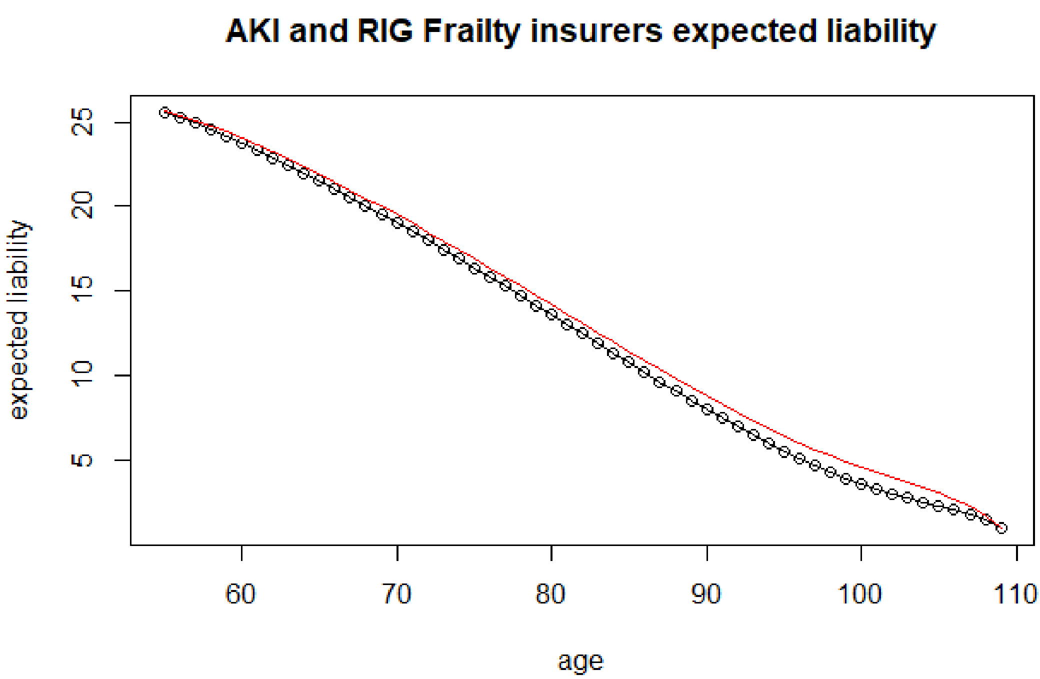
3. The reciprocal inverse Gaussian closely represents the AKI 2010 graduated rates between ages 55-80 beyond which there is longevity risk.

3.2.1 Deferred Life Annuity business

These are annuities which commence in m (say) years' time, provided that the annuitant is then active. Thus the present value of amount b payable for a future lifetime T_{x+t}

$$m|\bar{a}_x = \frac{D_{x+m}}{D_x} * a_{x+m}$$

Where $\frac{D_{x+m}}{D_x}$ is a pure endowment factor and a_{x+m} is an annuity factor at age x+m



KEY:

RED: RIG Frailty Annuity Expected Liability

BLACK: AKI 2010 graduated q-rates Expected Liability

Fig. 3. AKI and RIG Frailty insurers expected liability

Discussion

1. When heterogeneity is disregarded the expected liability is underestimated.

2. The Reciprocal Inverse Gaussian frailty is a close estimate of the insurer liability with a slight increase due to observed heterogeneity.

4. CONCLUSION

The proposed special cases of the GIG frailty show an increase in the insurers expected liability when unobserved heterogeneity is accounted for. That is, assuming the insureds to be homogeneous with respect to observed (underwriting) risk factors only could lead to an underestimation of future liability.
The RIG model as proposed in estimating future liability by directly adjusting the AKI mortality rates shows an increase in longevity risk more at extreme ages (beyond 80). The extent of heterogeneity of the insured group determines the level of risk. The GIG frailties should be considered for multivariate cases where the insureds are clustered in groups.

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APPENDIX1A

WINBUGS CODE

```
MODEL<-FUNCTION(){
FOR(I IN 1:55){
S[I] ~ DEXP(LAMBDA)    }
LAMBDA ~ DEXP(0.09)    }
WRITE.MODEL(MODEL,"MODEL.TXT")
INIT<-FUNCTION(){LIST(ALPHA=DEXP(0.9))}
DATA=LIST(S=GIG_TABLE$S.X._AKI.MALE.I.2.)
BUGS=BUGS(DATA=DATA,INITS=INIT,PARAMETERS.TO.SAVE=C("LAMBDA"),MODEL.
FILE="MODEL.TXT",BUGS.DIRECTORY="C:/USERS/DELL/DOCUMENTS/R/WIN-
LIBRARY/3.5/R2WINBUGS",N.CHAINS=1,N.ITER=100000,N.BURNIN=100,CODAPKG=TR
UE,DEBUG=T)
```

NODE STATISTICS

NODE	MEAN	SD	MC ERROR	2.5%	MEDIAN	97.5%	START	SAMPLE
DEVIANCE	100.8	1.396	0.04448	99.8	100.3	104.8	3	1010
LAMBDA	1.115	0.1478	0.004396	0.8378	1.105	1.422	3	1010

CASE 1 R-CODE:

```
PAR(MFROW=C(2,2))
AGE=56:1
SURVIVAL_FUNCTION=SEQ(0,1,1/55)
LAMBDA=1.115# WINBUGS ESTIMATE
S1 <-(1+5*LAMBDA*AGE)^(-1/5)
S2 <-(1+30*LAMBDA*AGE)^(-1/30)
S3 <-(1+60*LAMBDA*AGE)^(-1/60)
S4 <- EXP(-LAMBDA*AGE)
PLOT(AGE,SURVIVAL_FUNCTION,TYPE="N",MAIN="GAMMA-EXPONENTIAL MODEL")
LINES(AGE,S1,COL="BLACK")
LINES(AGE,S2,COL="BLUE")
LINES(AGE,S3,COL="GREEN")
LINES(AGE,S4,COL="RED")
```

CASE 2 R-CODE:

```
S1 <-EXP((1-(1+2*5*LAMBDA*AGE)^(0.5))/5)
S2 <-EXP((1-(1+2*30*LAMBDA*AGE)^(0.5))/30)
S3 <-EXP((1-(1+2*60*LAMBDA*AGE)^(0.5))/60)
PLOT(AGE,SURVIVAL_FUNCTION,TYPE="N",MAIN="IGAUSSIAN-EXPONENTIAL
MODEL")
LINES(AGE,S1,COL="BLACK")
LINES(AGE,S2,COL="BLUE")
LINES(AGE,S3,COL="GREEN")
```

```
LINES(AGE,S4,COL="RED")
```

CASE 3 R-CODE:

```
BETA=0.9
S1<-((1+2*BETA*LAMBDA*AGE)^(-0.5))*EXP((1-BETA)*(1-
(1+2*BETA*LAMBDA*AGE)^(0.5))/BETA)
PLOT(AGE,SURVIVAL_FUNCTION,TYPE="N",MAIN="R.I.G -EXPONENTIAL MODEL")
LINES(AGE,S1,COL="BLACK")
LINES(AGE,S4,COL="RED")
```

CASE 4 R-CODE:

```
S1 <- EXP(-SQRT(2*LAMBDA*AGE))
PLOT(AGE,S1,MAIN="LEVY-EXPONENTIAL MODEL",TYPE="L")
LINES(AGE,S4,COL="RED")
```

CASE 5 R-CODE:

```
PAR(MFROW=C(1,2))
A=0.05
N=2.5
S1 <- BESSELK(SQRT(2*A*N*LAMBDA*AGE+A^2),0)/(N*BESSELK(1/N,1))
PLOT(AGE,S1,YLAB="SURVIVALFUNCTION",TYPE="O",MAIN="HARMONIC-
EXPONENTIAL MODEL")
LINES(AGE,S4,COL="RED")
```

CASE 6 R-CODE:

```
VARPHI=10
THETA=0.03
S1<-SQRT((VARPHI^2)/(VARPHI+2*LAMBDA*AGE)*THETA)*(BESSELK
(SQRT((2*LAMBDA*AGE+VARPHI)*THETA),1)/BESSELK(SQRT(THETA*VARPHI),2))
PLOT(AGE,S1,YLAB="SURVIVALFUNCTION",TYPE="O",MAIN="POSITIVE
HYPERBOLIC-EXPONENTIAL MODEL")
LINES(AGE,S4,COL="RED")
```

COMPARISON GRAPHS 1 R-CODE

```
GIG_TABLE=READ.CSV("C:/USERS/DELL/DESKTOP/PROJECTS/GIG_BOOK1.CSV",HE
ADER=T)
PLOT(AGE..X.,S.X._AKI.MALE.I.2.,TYPE="O",MAIN="AKI AND FRAILTY SURVIVAL
FUNCTIONS", YLAB="SURVIVAL_FUNCTION", XLAB="AGE")
BETA=0.9
RIG_SX<-((1+2*BETA*BETA*GIG_TABLE$HX_AKI)^(-0.5))*EXP((1-BETA*BETA)*(1-
(1+2*BETA*BETA*GIG_TABLE$HX_AKI)^(0.5))/(BETA*BETA))
LINES(AGE..X.,RIG_SX,COL="GREEN",TYPE="O")
```

COMPARISON GRAPHS 2 R-CODE

```
PLOT(AGE..X.,AX_AKI,TYPE="O",MAIN="AKI AND RIG FRAILTY INSURERS EXPECTED
LIABILITY", YLAB="EXPECTED LIABILITY", XLAB="AGE")
LINES(AGE..X.,AX_RIG,COL="RED")
```

TABLE1.0: ASSOCIATION OF KENYA INSURER'S LIFETABLE								
AKI male I=2%								
Age (x)	lx	dx	px	qx	hx	Dx	Nx	ax
55	100000	373	0.99627	0.00373	0.00374	33650.42	860086	25.56
56	99627	375	0.99624	0.00376	0.00377	32867.49	829818.8	25.25
57	99252	382	0.99615	0.00385	0.00386	32101.78	799620.6	24.91
58	98870	391	0.99604	0.00396	0.00396	31351.19	769549.4	24.55
59	98479	403	0.99591	0.00409	0.0041	30614.87	739679.9	24.16
60	98076	415	0.99577	0.00423	0.00424	29891.88	710096.8	23.76
61	97661	428	0.99562	0.00438	0.00439	29181.77	680886.2	23.33
62	97233	442	0.99545	0.00455	0.00456	28484.15	652129.2	22.89
63	96791	457	0.99528	0.00472	0.00473	27798.66	623893.4	22.44
64	96334	473	0.99509	0.00491	0.00492	27124.93	596229.8	21.98
65	95861	491	0.99488	0.00512	0.00513	26462.49	569172.7	21.51
66	95370	511	0.99464	0.00536	0.00537	25810.76	542710.2	21.03
67	94859	533	0.99438	0.00562	0.00564	25169.11	516899.4	20.54
68	94326	559	0.99408	0.00592	0.00594	24536.85	491730.3	20.04
69	93767	587	0.99375	0.00626	0.00627	23913.3	467193.4	19.54
70	93181	617	0.99337	0.00663	0.00665	23297.77	443280.1	19.03
71	92563	652	0.99296	0.00704	0.00707	22689.61	419982.4	18.51
72	91912	690	0.9925	0.0075	0.00753	22088.09	397292.8	17.99
73	91222	732	0.99198	0.00802	0.00806	21492.47	375204.7	17.46
74	90490	778	0.9914	0.0086	0.00864	20901.99	353712.2	16.92
75	89711	830	0.99075	0.00925	0.0093	20315.86	332810.2	16.38
76	88881	888	0.99001	0.00999	0.01004	19733.19	312494.4	15.84
77	87994	952	0.98918	0.01082	0.01087	19153.03	292761.2	15.29
78	87042	1023	0.98825	0.01175	0.01182	18574.39	273608.1	14.73
79	86019	1103	0.98718	0.01282	0.0129	17996.14	255033.8	14.17
80	84916	1192	0.98596	0.01404	0.01414	17417.07	237037.6	13.61
81	83724	1292	0.98456	0.01544	0.01556	16835.82	219620.5	13.04
82	82431	1405	0.98296	0.01704	0.01719	16250.92	202784.7	12.48
83	81026	1531	0.9811	0.0189	0.01908	15660.76	186533.8	11.91
84	79495	1673	0.97895	0.02105	0.02127	15063.55	170873	11.34
85	77822	1833	0.97645	0.02356	0.02384	14457.33	155809.5	10.78
86	75989	2012	0.97352	0.02648	0.02684	13839.99	141352.2	10.21
87	73977	2213	0.97008	0.02992	0.03038	13209.28	127512.2	9.65
88	71763	2437	0.96604	0.03396	0.03455	12562.81	114302.9	9.1
89	69326	2685	0.96127	0.03873	0.0395	11898.19	101740.1	8.55
90	66641	2957	0.95562	0.04438	0.04539	11213.06	89841.91	8.01
91	63683	3252	0.94894	0.05106	0.05241	10505.34	78628.85	7.48
92	60432	3564	0.94102	0.05898	0.06079	9773.46	68123.5	6.97
93	56867	3887	0.93164	0.06836	0.07081	9016.68	58350.04	6.47
94	52980	4209	0.92056	0.07944	0.08277	8235.61	49333.36	5.99
95	48771	4510	0.90752	0.09248	0.09704	7432.75	41097.75	5.53
96	44261	4769	0.89226	0.10774	0.114	6613.14	33665	5.09
97	39493	4955	0.87453	0.12548	0.13407	5784.93	27051.86	4.68
98	34537	5039	0.85411	0.14589	0.15769	4959.87	21266.92	4.29
99	29499	4989	0.83088	0.16912	0.18527	4153.23	16307.05	3.93
100	24510	4785	0.80479	0.19521	0.21718	3383.19	12153.82	3.59
101	19725	4421	0.77588	0.22412	0.25375	2669.36	8770.64	3.29
102	15305	3913	0.74431	0.25569	0.29529	2030.5	6101.27	3
104	8092	2642	0.67344	0.32656	0.39536	1031.84	2589.07	2.51
105	5449	1983	0.63605	0.36395	0.45247	681.25	1557.23	2.29
106	3466	1399	0.59625	0.40375	0.5171	424.82	875.98	2.06
107	2067	920	0.55481	0.44519	0.58913	248.33	451.17	1.82
108	1147	560	0.51172	0.48828	0.66998	135.07	202.84	1.5
109	587	587	0.46686	0.53314	0.76172	67.76	67.76	1

TABLE2.0: RECIPROCAL INVERSE GAUSSIAN LIFETABLE								
RIG FRAILTY								
Age (x)	lx	dx	px	qx	µx	Dx	Nx	ax
55	1E+08	371910	0.996281	0.003719	0.003726	33650425	878248787.2	26.1
56	99628090	373486	0.996251	0.003749	0.003756	32867917	844598362.2	25.7
57	99254604	380919	0.996162	0.003838	0.003845	32102649	811730444.9	25.29
58	98873685	389226	0.996063	0.003937	0.003944	31352398	779627795.8	24.87
59	98484459	401324	0.995925	0.004075	0.004083	30616643	748275398.2	24.44
60	98083135	413244	0.995787	0.004213	0.004222	29894000	717658755.4	24.01
61	97669891	425968	0.995639	0.004361	0.004371	29184363	687764755.3	23.57
62	97243923	440418	0.995471	0.004529	0.004539	28487335	658580391.9	23.12
63	96803505	454657	0.995303	0.004697	0.004708	27802270	630093057	22.66
64	96348848	470568	0.995116	0.004884	0.004896	27129109	602290786.6	22.2
65	95878280	488107	0.994909	0.005091	0.005104	26467265	575161677.3	21.73
66	95390174	508153	0.994673	0.005327	0.005341	25816199	548694411.9	21.25
67	94882021	530647	0.994407	0.005593	0.005608	25175170	522878212.7	20.77
68	94351374	555503	0.994112	0.005888	0.005905	24543503	497703042.4	20.28
69	93795871	582622	0.993788	0.006212	0.006231	23920589	473159539.3	19.78
70	93213249	613753	0.993416	0.006584	0.006606	23305886	449238950.3	19.28
71	92599495	647817	0.993004	0.006996	0.00702	22698462	425933064.2	18.76
72	91951678	684672	0.992554	0.007446	0.007474	22097711	403234602.6	18.25
73	91267006	726832	0.992036	0.007964	0.007996	21503109	381136891.4	17.72
74	90540174	772271	0.99147	0.00853	0.008566	20913591	359633781.9	17.2
75	89767903	823369	0.990828	0.009172	0.009215	20328634	338720190.7	16.66
76	88944533	879777	0.990109	0.009891	0.009941	19747231	318391556.7	16.12
77	88064756	941932	0.989304	0.010696	0.010754	19168535	298644325.7	15.58
78	87122824	1011888	0.988386	0.011615	0.011682	18591676	279475791.2	15.03
79	86110936	1089786	0.987344	0.012656	0.012736	18015435	260884114.9	14.48
80	85021151	1177288	0.986153	0.013847	0.013944	17438665	242868680.3	13.93
81	83843863	1274930	0.984794	0.015206	0.015323	16859992	225430015.1	13.37
82	82568933	1383781	0.983241	0.016759	0.016901	16278058	208570023	12.81
83	81185152	1506041	0.981449	0.018551	0.018725	15691424	192291965.1	12.25
84	79679111	1642553	0.979385	0.020615	0.02083	15098370	176600541.3	11.7
85	78036558	1796402	0.97698	0.02302	0.023289	14497179	161502171.8	11.14
86	76240156	1967423	0.974194	0.025806	0.026144	13885739	147004992.6	10.59
87	74272733	2158522	0.970938	0.029062	0.029493	13262166	133119253.4	10.04
88	72114212	2369435	0.967143	0.032857	0.033409	12624255	119857087.3	9.49
89	69744777	2601745	0.962696	0.037304	0.038017	11970062	107232832.6	8.96
90	67143031	2854667	0.957484	0.042516	0.043446	11297582	95262770.61	8.43
91	64288365	3125668	0.951381	0.04862	0.049841	10605148	83965189.01	7.92
92	61162697	3410102	0.944245	0.055755	0.057369	9891697	73360040.62	7.42
93	57752595	3700723	0.935921	0.064079	0.066224	9157049	63468343.19	6.93
94	54051872	3985380	0.926268	0.073733	0.076592	8402231	54311294.37	6.46
95	50066493	4249093	0.915131	0.084869	0.088688	7630111	45909063.77	6.02
96	45817399	4471613	0.902404	0.097596	0.102693	6845638	38278952.86	5.59
97	41345786	4630439	0.888007	0.111993	0.118776	6056401	31433314.6	5.19
98	36715348	4702667	0.871916	0.128085	0.137063	5272673	25376914	4.81
99	32012681	4667791	0.854189	0.145811	0.157602	4507181	20104241.33	4.46
100	27344889	4512724	0.83497	0.16503	0.180359	3774496	15597059.93	4.13
101	22832165	4236211	0.814463	0.185537	0.205226	3089796	11822563.73	3.83
102	18595954	3851092	0.792907	0.207093	0.232049	2467181	8732768.165	3.54
103	14744862	3382810	0.770577	0.229423	0.260616	1917887	6265587.614	3.27
104	11362051	2871363	0.747285	0.252715	0.291309	1448902	4347700.625	3
105	8490688	2338887	0.724535	0.275465	0.322225	1061512	2898799.054	2.73
106	6151801	1838543	0.701138	0.298863	0.355051	754022.1	1837287.171	2.44
107	4313258	1390785	0.677556	0.322444	0.389263	518307	1083265.1	2.09
108	2922474	1011977	0.653726	0.346274	0.425067	344296.1	564958.0909	1.64