Original Research Article

Second order rotatable designs of second type using pairwise balanced designs

Abstract

In this paper, second order rotatable designs of second type using pairwise balanced designs is suggested. This design is compared with second order rotatable designs of first type using pairwise balanced designs (Tyagi, 1964) on the basis of efficiency.

Keywords: Response surface methodology, Pairwise balanced designs, Rotatability, Orthogonality, Efficiency.

1. Introduction

Response surface designs is a collection of mathematical and statistical techniques useful for analyzing problems where several independent variables influence a dependent variable or response. Box and Hunter (1957) introduced designs having spherical variance function are called rotatable designs. Das and Narasimham (1962) constructed rotatable designs using balanced incomplete block designs (BIBD). Raghavarao (1963) constructed SORD using symmetrical unequal block arrangements (SUBA) with two unequal block sizes. Draper and Guttman (1988) suggested an index of rotatability. Khuri (1988) introduced a measure of rotatability for response surface designs. Draper and Pukelshein (1990) developed another look at rotatability. Park et al. (1993) introduced new measure of rotatability for second order response surface designs. Das et al. (1999) developed modified response surface designs. Kim (2002) introduced extended central composite designs (CCD) with the axial points are indicated by two numbers. Kim and Ko (2004) developed slope rotatability of second type of CCD. Victorbabu and Vasundharadevi (2005) suggested modified second order response surface designs using BIBD. Victorbabu (2006) constructed modified SORD and second order slope rotatable designs using a pair of BIBD. Victorbabu et al. (2006) studied modified second order response surface designs using pairwise balanced designs (PBD). Victorbabu (2007) suggested a review on second order rotatable designs. Victorbabu et al. (2008) suggested modified second order response surface designs using CCD. Victorbabu and Vasundharadevi (2008) studied second order response surface designs using SUBA with two unequal block sizes. Victorbabu (2009) constructed modified SORD using a pair SUBA with two unequal block sizes. Park and Park (2010) suggested the extension of CCD for second order response surface models. Victorbabu and Surekha (2013) suggested measure of rotatability for second order response surface designs using incomplete block designs. Victorbabu and Surekha (2015) developed measure of rotatability for second order response surface designs using BIBD. Kim (2019) suggested modified slope rotatability using extended CCD. Chiranjeevi et al. (2021) extended the work of Kim (2002) and suggested second order rotatable designs of second type using CCD for $9 \le v \le 17$ (v: number of factors). Chiranjeevi and Victorbabu (2021) developed SORD second type using BIBD

In this paper, second order rotatable designs of second type using pairwise balanced designs is suggested. This design is compared with second order rotatable designs of first type using pairwise balanced designs (Tyagi, 1964) on the basis of efficiency.

2. Stipulations and formulas for second order rotatable designs

Suppose we want use the second order polynomial response surface design D = ((x_{iu})) to fit the surface,

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i<j} \sum b_{ij} x_{iu} x_{ju} + \varphi_u \quad (2.1)$$

where x_{iu} represents the level of i^{th} factor (i=1,2,…,v) in the u^{th} run (u=1,2,…,N) of the experiment and φ_u are uncorrelated random error with mean zero and variance σ^2. Then ‘D’ is said to be second order rotatable designs (SORD) if the variance of $Y_u(x_1, x_2, \dots, x_v)$ with respect to each of independent variable (x_i) is only a function of the distance ($d^2 = \sum_{i=1}^{v} x_i^2$) of the point ($x_1, x_2, \dots, x_v$) from the origin (center) of the design. Such a spherical variance function for estimation of responses in the second order polynomial model is achieved if the design points satisfy the following conditions (cf. Das and Giri 1999).

All odd order moments are must be zero. In their words when at least one odd power x’s equal to zero.

$$1.\quad \sum x_{iu}=0, \sum x_{iu}x_{ju}=0, \sum x_{iu}x_{ju}^2=0, \sum x_{iu}x_{ju}x_{ku}=0,$$
$$\sum x_{iu}^3=0, \sum x_{iu}x_{ju}^3=0, \sum x_{iu}x_{ju}x_{ku}^2=0, \sum x_{iu}x_{ju}x_{ku}x_{lu}=0.$$

for $i\neq j\neq k\neq l$; (2.2)

2. (i) $\sum x_{iu}^2=\text{constant}=N\mu_2$ (ii) $\sum x_{iu}^4=\text{constant}=cN\mu_4$ for all i (2.3)

3. $\sum x_{iu}^2 x_{ju}^2=\text{constant}=N\mu_4;\ \text{for all}\ i\neq j$ (2.4)

4. $\frac{\mu_4}{\mu_2^2}>\frac{v}{(c+v-1)}$ (2.5)

5. $\sum x_{iu}^4=c\sum x_{iu}^2 x_{ju}^2$ (2.6)

where c, μ_4 and μ_2 are constants.

The variances and covariances of the estimated parameters are

$$V(\hat{b}_0)=\frac{\mu_4(c+v-1)\sigma^2}{N[\mu_4(c+v-1)-v\mu_2^2]},$$

$$V(\hat{b}_i)=\frac{\sigma^2}{N\mu_2},$$

$$V(\hat{b}_{ij})=\frac{\sigma^2}{N\mu_4},$$

$$V(\hat{b}_{ii})=\frac{\sigma^2}{(c-1)N\mu_4}\left[\frac{\mu_4(c+v-2)-(v-1)\mu_2^2}{\mu_4(c+v-1)-v\mu_2^2}\right],$$

$$Cov(\hat{b}_0,\hat{b}_{ii})=\frac{-\mu_2\sigma^2}{N[\mu_4(c+v-1)-v\mu_2^2]},$$

$$Cov(\hat{b}_{ii},\hat{b}_{jj})=\frac{(\mu_2^2-\mu_4)\sigma^2}{(c-1)N\mu_4[\mu_4(c+v-1)-v\mu_2^2]}$$ and other covariances vanish. (2.7)

The variance of the estimated response at the point (x_{10}, x_{20}, …, x_{v0}) is

$$V(\hat{y}_0)=V(\hat{b}_0)+\left[V(\hat{b}_i)+2Cov(\hat{b}_0,\hat{b}_{ii})\right]d^2+V(\hat{b}_{ii})d^4+ \sum x_{i0}^2x_{j0}^2\left[V(\hat{b}_{ij})+2Cov(\hat{b}_{ii},\hat{b}_{jj})-2V(\hat{b}_{ii})\right] \quad (2.8)$$

The coefficient of $\sum x_{i0}^2x_{j0}^2$ in the above equation (2.8) is simplified to $(c\text{-}3)\sigma^2/(c\text{-}1)N\lambda_4$.

A second order response surface design D is said to be SORD, if in this design c=3 and all the other conditions (2.2) to (2.7) hold.

3. SORD of first type using pairwise balanced designs (cf. Tyagi (1964)).

Let (v, b, r, k_1, k_2, .., k_p, λ) denote parameters of PBD, k=sup(k_1,k_2,..,k_p) and $2^{t(k)}$ denote a fractional replication of 2^k in +1 or -1 levels in which no interaction with less than five factors is confounded. [1-(v, b, r, k_1, k_2, . . . , k_p, λ)] denote the design points generated from transpose of the incidence matrix of PBD. Let $[1\text{-}(v,b,r,k_1,k_2,...,k_p,\lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from PBD by "multiplication" (cf. Raghavarao (1971), pp 298-300), $(\pm a, 0, 0, \ldots,0)2^1$ denote the design points generated from (±a, 0, 0, …, 0) point set. Let U denote the union of the design points generated from different sets of points, n_0 denote the number of central points. The method of construction of SORD of first type using PBD is given in the following result

Result: The design points, $[1\text{-}(v,b,r,k_1,k_2,...,k_p,\lambda)]2^{t(k)}U(a,0,...,0)2^1U(n_0)$ will give a v-dimensional SORD of first type using PBD in $N=b2^{t(k)}+2v+n_0$ design points, with

$$a^4=\frac{2^{t(k)}(3\lambda\text{-}r)}{2}.$$

The condition for the design becomes an orthogonal design.

From equation 2 (i) of (2.3) and (3) of (2.4), we have

$$\sum x_{iu}^2=r2^{t(k)}+2a^2=N\mu_2$$

$$\sum x_{iu}^2x_{ju}^2=\lambda2^{t(k)}=N\mu_4$$

For the convenience N is replaced by M

By using the orthogonality condition we have

$$\mu_2^2=\mu_4$$

$$\left(\frac{r2^{t(k)}+2a^2}{M}\right)^2=\frac{\lambda 2^{t(k)}}{M}$$

then we can obtain

$$a^2=\left(\frac{\sqrt{\lambda 2^{t(k)}M}-r2^{t(k)}}{2}\right)(\text{for orthogonality})$$

(3.1)

and the condition for the design become rotatability .

From equation 2 (ii) of (2.3) and 3 of (2.4), we have

$$\sum x_{iu}^4=r2^{t(k)}+2a^4=3N\mu_4$$

$$\sum x_{iu}^2 x_{ju}^2=\lambda 2^{t(k)}=N\mu_4$$

For the convenience N is replace by M

Then the rotatability condition equation (2.6), we have

$$\sum x_{iu}^4=c\sum x_{iu}^2 x_{ju}^2$$

$$\Rightarrow r2^{t(k)}+2a^4=3(\lambda 2^{t(k)})$$

$$\Rightarrow a^4=\left(\frac{(3\lambda-r)2^{t(k)}}{2}\right)(\text{for rotatability})$$

(3.2)

3.1 Proposed method of SORD of second type using pairwise balanced design

Let $(v,b,r,k_1,k_2,..,k_p,\lambda)$ denote a parameters of PBD, $k=\sup(k_1,k_2,..,k_p)$ and $2^{t(k)}$ denote a fractional replication of 2^k in +1 or -1 levels in which no interaction with less than five factors are confounded. $[1\text{-}(v,b,r,k_1,k_2,...,k_p,\lambda)]$ denote the design points generated from transpose of the incidence matrix of PBD. Let $[1\text{-}(v,b,r,k_1,k_2,...,k_p,\lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from PBD by "multiplication" (cf. Raghavarao (1971), pp 298-300). We use the additional set of points like $(\pm a_1,0,...,0),(0,\pm a_1,0,...,0),...,(0,0,...,\pm a_1)$ and $(\pm a_2,0,...,0),(0,\pm a_2,0,...,0),...,(0,0,...,\pm a_2)$ are two sets of axial points. Here $(a_1,0,0,...,0)2^1U(a_2,0,0,...,0)2^1$ denote the 4v design points generated from $(a_1,0,0,...,0)U(a_2,0,0,...,0)$ point set. Let U denote the union of the design points generated from different sets of points, and (n_0) denote the number of central points. The method of construction of SORD of second type using PBD is given in the following theorem.

Theorem (3.1): The design points,

$[1\text{-}(v,b,r,k,,\lambda)]2^{t(k)}U(a_1,0,...,0)2^1U(a_2,0,...,0)2^1U(n_0)$ will give a v-dimensional SORD of second type using PBD in $N=b2^{t(k)}+4v+n_0$ design points, with

$$a_1^4+a_2^4=\frac{2^{t(k)}(3\lambda\text{-}r)}{2}\ (\text{for rotatability}).$$

Proof: For the design points generated from second order rotatable designs of second type using PBD, simple symmetry conditions (2.2) are true. Further, conditions (2.3) and (2.4) are true as follows:

$$\sum x_{iu}^2=r2^{t(k)}+2(a_1^2+a_2^2)=N\mu_2 \qquad (3.3)$$

$$\sum x_{iu}^4=r2^{t(k)}+2(a_1^4+a_2^4)=cN\mu_4 \qquad (3.4)$$

$$\sum x_{iu}^2x_{ju}^2=\lambda 2^{t(k)}=N\mu_4 \qquad (3.5)$$

Solving equations (3.4) and (3.5), we get $a_1^4+a_2^4=\frac{2^{t(k)}(3\lambda\text{-}r)}{2}(\text{for rotatability})$.

Example 3.1. We illustrate the theorem (3.1) to obtain SORD of second type using PBD with parameters $(v=9,b=11,r=5,k_1=5,k_2=4,k_3=3,\lambda=2)$. The design points

$[1\text{-}(9,11,5,5,4,3,2)]2^{t(5)}U(a_1,0,0,...,0)2^1U(a_2,0,0,...,0)2^1U(n_0=1)$ will give a v-dimensional SORD of second type using PBD in N=213 design points with one central point. From (3.3), (3.4) and (3.5) we have

$$\sum x_{iu}^2=80+2(a_1^2+a_2^2)=N\mu_2 \tag{3.6}$$

$$\sum x_{iu}^4=80+2(a_1^4+a_2^4)=cN\mu_4 \tag{3.7}$$

$$\sum x_{iu}^2x_{ju}^2=32=N\mu_4 \tag{3.8}$$

From (3.7) and (3.8), we can obtain the rotatability value $a_1^4+a_2^4=8,$ here we assume for an arbitrary value a_1=1, then we get a_2=1.6266 and c=3 (for rotatability).

The non-singularity condition (2.5), we have

$$\frac{0.1502}{0.1502}>\frac{9}{3+9-1} \Rightarrow 1.0000>0.8182.$$

Hence the non singularity condition is also satisfied.

The variances and covariances of the estimated parameters are

$$V(\hat{b}_0)=0.0258\sigma^2$$

$$V(\hat{b}_i)=0.0121\sigma^2$$

$$V(\hat{b}_{ij})=0.0313\sigma^2$$

$$V(\hat{b}_{ii})=0.0156\sigma^2$$

$$Cov(\hat{b}_0,\hat{b}_{ii})= -0.0061\sigma^2$$

$$Cov(\hat{b}_{ii},\hat{b}_{jj})=0 \tag{3.9}$$

The variance of the estimated response at the point $(x_{10}, x_{20},\ldots,x_{v0})$ is

$$V(\hat{Y})=0.0258\sigma^2+d^2(-0.0001\sigma^2)+d^4(0.0156\sigma^2) \quad (3.10)$$

Table 1 gives the values of the variance of the estimated responses for different factors by using PBD

Table 1 The variance of estimated response for different factors

$(v, b, r, k_1,k_2,\ldots, k_p, \lambda)$	n_0	N	$a_1^4+a_2^4$	a_1	a_2	$V(\hat{Y})$
(9,11,5,5,4,3,2)	1	213	8	1	1.6266	$(0.0258-0.0001\,d^2+0.0156\,d^4)\,\sigma^2$
(10,11,5,5,4,2)	1	217	8	1	1.6266	$(0.0276+0.0002\,d^2+0.0156\,d^4)\,\sigma^2$
(13,15,7,7,6,5,3)	34	1046	64	1	2.8173	$(0.0072+0.0026\,d^4)\,\sigma^2$
(14,15,7,7,6,3)	30	1046	64	1	2.8173	$(0.0076+0.0026\,d^4)\,\sigma^2$

4. Study of orthogonality in SORD of second type using PBD (cf. Kim (2002).

An orthogonal design is one in which the terms in the fitted model are uncorrelated with one another and thus the parameter estimates are uncorrelated. In this case, the variance of the predicated response at any point x in the experimental region, is expressible as a weighted sum of the variance of the parameter estimates in the model. For second order moments $\sum x_{iu}^2$ and $\sum x_{iu}^2 x_{ju}^2$ is impossible to obtain. This is because the moments $\sum x_{iu}^2$ and $\sum x_{iu}^2 x_{ju}^2$ are necessarily positive. Hence, we consider the model with the pure quadratic terms correlated for their means. In regard to orthogonality, this model is often used for the sake of simplicity in the calculation. A design is said to be orthogonal we shall investigate the restriction $\left(\sum x_{iu}^2\right)^2=N\sum x_{iu}^2 x_{ju}^2$ i.e $\left(N\mu_2^2\right)^2=N\left(N\mu_4\right)$ i.e $\mu_2^2=\mu_4$ to get SORD of second type using PBD.

$$a_1^2+a_2^2=\frac{\sqrt{N(\lambda 2^{t(k)})}-r2^{t(k)}}{2} \quad (4.1)$$

It must be established the equation (4.1) makes SORD of second type using PBD an orthogonal system. However $N=2^{t(v)}+4v+n_0$, the value of (4.1) depends on v, n_0 and the design points of SORD of second type. The following table 1 gives the values of orthogonality of second order response surface methodology using various parameters of SORD of second type and n_0, the value of '$a_1^2+a_2^2$' makes orthogonal second order response surface designs by using SORD of second type using PBD.

Let $(v,b,r,k_1,k_2,..,k_p,\lambda)$ denote a parameters of PBD, $k=\sup(k_1,k_2,..,k_p)$ and $2^{t(k)}$ denote a fractional replication of 2^k in +1 or -1 levels in which no interaction with less than five factors are confounded. $[1\text{-}(v,b,r,k_1,k_2,...,k_p,\lambda)]$ denote the design points generated from transpose of the incidence matrix of PBD. Let $[1\text{-}(v,b,r,k_1,k_2,...,k_p,\lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from PBD by "multiplication" (cf. Raghavarao (1971), pp 298-300). We use the additional set of points like $(\pm a_1,0,...,0),(0,\pm a_1,0,...,0),...,(0,0,...,\pm a_1)$ and $(\pm a_2,0,...,0),(0,\pm a_2,0,...,0),...,(0,0,...,\pm a_2)$ of two sets of axial points. Here $(a_1,0,0,...,0)2^1U(a_2,0,0,...,0)2^1$ denote the 4v design points generated from $(a_1,0,0,...,0)U(a_2,0,0,...,0)$ point set. Let U denote the union of the design points generated from different sets of points, (n_0) denote the number of central points. The method of study of orthogonality of SORD of second type using PBD is given in the following theorem.

Theorem (4.1): The design points,

$[1\text{-}(v,b,r,k,,\lambda)]2^{t(k)}U(a_1,0,...,0)2^1U(a_2,0,...,0)2^1U(n_0)$ will give a v-dimensional SORD of second type using PBD in $N=b2^{t(k)}+4v+n_0$ design points, with

$$a_1^2+a_2^2=\frac{\sqrt{N(\lambda 2^{t(k)})\text{-}r2^{t(k)}}}{2}\text{(for orthogonality).}$$

Proof: For the design points generated from second order rotatable designs of second type using PBD, simple symmetry conditions (2.2) are true. Further, conditions (2.3) and (2.4) are true as follows

$$\sum x_{iu}^2=r2^{t(k)}+2(a_1^2+a_2^2)=N\mu_2 \qquad (4.2)$$

$$\sum x_{iu}^4=r2^{t(k)}+2(a_1^4+a_2^4)=cN\mu_4 \qquad (4.3)$$

$$\sum x_{iu}^2x_{ju}^2=\lambda 2^{t(k)}=N\mu_4 \qquad (4.4)$$

Solving equations (4.2) and (4.4) using $\mu_2^2=\mu_4$

$$\left(\frac{r2^{t(k)}+2(a_1^2+a_2^2)}{N}\right)^2=\frac{\lambda 2^{t(k)}}{N}$$

then we can obtain

$$a_1^2+a_2^2=\frac{\sqrt{N(\lambda 2^{t(k)})}-r2^{t(k)}}{2}\text{(for orthogonality).}$$

Example 4.1. We illustrate the theorem (4.1) second order rotatable designs of second type using PBD with parameters $(v=9,b=11,r=5,k_1=5,k_2=4,k_3=3,\lambda=2)$. The design points $[1-(9,11,5,5,4,3,2)]2^{t(5)}U(a_1,0,0,...,0)2^1U(a_2,0,0,...,0)2^1U(n_0=1)$ will give a v-dimensional SORD of second type using PBD in N=213 design points with one central point. From (3.3), (3.4) and (3.5) we have

$$\sum x_{iu}^2=80+2(a_1^2+a_2^2)=N\mu_2 \quad (4.5)$$

$$\sum x_{iu}^4=80+2(a_1^4+a_2^4)=cN\mu_4 \quad (4.6)$$

$$\sum x_{iu}^2x_{ju}^2=32=N\mu_4 \quad (4.7)$$

From (4.5) and (4.7), using $\mu_2^2=\mu_4$, we can obtain the orthogonality value

$$\Rightarrow a_1^2+a_2^2=\frac{\sqrt{(213)\times(32)}-80}{2}$$

$$=1.2795.$$

The values of orthogonality of SORD of second type using PBD for $9\leq v\leq 14$ with central points are given in the following table 2

Table 2: values of orthogonality of SORD of second type using PBD

(9,11,5,5,4,3,2)	(10,11,5,5,4,2)	(13,15,7,7,6,5,3)	(14,15,7,7,6,3)

$(n_0=1)$	N	$a_1^2+a_2^2$	$(n_0=1)$	N	$a_1^2+a_2^2$	$(n_0=34)$	N	$a_1^2+a_2^2$	$(n_0=30)$	N	$a_1^2+a_2^2$
n_0	213	1.2795	n_0	217	1.6653	n_0	1046	0.0714	n_0	1046	0.0714
n_0+1	214	1.3763	n_0+1	218	1.7612	n_0+1	1047	0.1785	n_0+1	1047	0.1785
n_0+2	215	1.4729	n_0+2	219	1.8569	n_0+2	1048	0.2855	n_0+2	1048	0.2855
n_0+3	216	1.5692	n_0+3	220	1.9524	n_0+3	1049	0.3925	n_0+3	1049	0.3925
n_0+4	217	1.6653	n_0+4	221	2.0476	n_0+4	1050	0.4994	n_0+4	1050	0.4994

5. Efficiency comparison for SORD of second type using PBD with SORD of first type using PBD

In this section, SORD of second type using PBD is used as the basis for estimating specific coefficient in the response surface model, SORD of second type using PBD is compared with SORD of first type using PBD. This comparison criterion is based on the precision at which the coefficient is estimated. It is consider that the numbers of experimental plots are required at same way.

For example in terms of estimating mixed quadratic coefficient b_{ij} $(i\neq j)$, two experimental designs, lets try to compare D_1 and D_2. The number of experimental plots required in D_1 and D_2 are M and N respectively. The relative efficiency of D_1 and D_2 is given by the following equation (see Myers (1976), section 7.2).

$$E\left(D_1/D_2\right)=\frac{\{Var(b_{ij})\,in\,D_2\}N}{\{Var(b_{ij})\,in\,D_1\}M} \tag{5.1}$$

Where $N=b2^{t(k)}+4v+n_0$ (Design points in SORD of second type using PBD)

$M=b2^{t(k)}+2v+m_0$ (Design points in SORD of first type using PBD)

In this case, in order to compare fairly, the experimental system should make the second product equal to value of $\frac{\sum x_{iu}^2}{N}$. It must be scaled and for this the following scaling criteria is used.

$$(i)\ \frac{1}{N}\sum x_{iu}=0$$

$$(ii)\ \frac{1}{N}\sum x_{iu}^{2}=1,\ (i=1,2,\ldots,v) \tag{5.2}$$

5.1 Comparison in mixed quadratic coefficient $b_{ij}\ (i \neq j)$

According to equation (2.7) $V(\hat{b}_{ij})=\frac{\sigma^2}{N\mu_4}$, but this before scaling equation (5.2) will be write in SORD of second type using PBD. From (ii) of (5.2) $\frac{(r2^{t(k)}+2a_1^2+2a_2^2)}{N}$ each time making equation (5.2) will be is equal to 1, i.e. (ii) =1, then the scaling factor g

$$g=\left\{\left(\frac{b2^{t(K)}+4v+n_0}{r2^{t(k)}+2a_1^2+2a_2^2}\right)\right\}^{1/2} \tag{5.3}$$

However the $V(b_{ij})$ is multiplied by with the scaling factor 'g' than the $V(b_{ij})=\frac{\sigma^2}{N\lambda_4}.\frac{1}{g^4}$ that is

$$V(b_{ij})=\frac{\sigma^2}{N\lambda_4}\left\{\left[\frac{r2^{t(k)}+2a_1^2+2a_2^2}{b2^{t(k)}+4v+n_0}\right]\right\}^2$$

According to equation (5.1) the relative efficiency SORD of second type using PBD versus SORD of first type using PBD in the mixed quadratic coefficient b_{ij} is obtained as follows

$$E\left(\frac{\text{SORD of second type using PBD}}{\text{SORD of first type using PBD}}\right)=\frac{\frac{\sigma^2}{N\lambda_4}\left(\frac{r2^{t(k)}+2a^2}{b2^{t(k)}+2v+m_0}\right)^2(b2^{t(k)}+2v+m_0)}{\frac{\sigma^2}{N\lambda_4}\left(\frac{r2^{t(k)}+2a_1^2+2a_2^2}{b2^{t(k)}+4v+n_0}\right)^2(b2^{t(k)}+4v+n_0)}$$

$$=\frac{(r2^{t(k)}+2a^2)^2(b2^{t(k)}+4v+n_0)}{(r2^{t(k)}+2a_1^2+2a_2^2)^2(b2^{t(k)}+2v+m_0)} \quad (5.4)$$

From equation (5.4) the condition that $E\left(\frac{\text{SORD of second type using PBD}}{\text{SORD of first type using PBD}}\right)>1$, than the SORD of second type using PBD is more efficient than SORD of first type using PBD

$$a_1^2+a_2^2<\frac{1}{2}\left\{(r2^{t(k)}+2a^2)\sqrt{\frac{b2^{t(k)}+4v+n_0}{b2^{t(k)}+2v+m_0}}-r2^{t(k)}\right\} \quad (5.5)$$

From the values of (3.1) and (4.1) substitute in (5.4) and then we get the value of greater than 1. From this orthogonal SORD of second type using PBD has the same degree of efficiency as orthogonal SORD of first type using PBD, and consider the efficiency of SORD of second type using PBD is giving the better efficiency than SORD of first type using PBD. Now, the efficiency comparison of SORD of second type using PBD versus SORD of first type using PBD with rotatability. Substituting the values of (3.2) into (5.5) and we evaluated that the SORD of second type using PBD will be more efficient than the previous SORD of first type using PBD with rotatability.

$$a_1^2+a_2^2<\frac{1}{2}\left\{\left(r2^{t(k)}+2\sqrt{\frac{(3\lambda-r)2^{t(k)}}{2}}\right)\sqrt{\frac{b2^{t(k)}+4v+n_0}{b2^{t(k)}+2v+m_0}}-r2^{t(k)}\right\} \quad (5.6)$$

For example, in SORD of first type using PBD when $(v=9,b=11,r=5,k_1=5,k_2=4,k_3=3,\lambda=2)$ and $m_0=1$ then we get M =195 then a=1.6818 and $n_0=1$, then the equation (5.6)

$$a_1^2+a_2^2<4.7599 \quad (5.7)$$

Among the rotatability of second type of PBD, it is easy to find an experimental plan that satisfies the equation (5.7). For example in SORD of second type using PBD with $a_1=1$, $a_2=1.6266$ satisfy the rotatability property equation (3.7), and $a_1^2+a_2^2=3.6458$ as it satisfies the equation (5.7) as well, it is more efficient than the rotatability SORD of first type using PBD.

Then the relative efficiency of SORD of second type using PBD versus SORD of first type using PBD equation (5.4) is as follows.

$$\frac{(80+2(2.8285))^2(176+36+1)}{(80+2(3.6458))^2(176+18+1)}=1.0518$$

5.2 Comparison in the pure quadratic coefficient b_{ii}

Now this time in terms of estimating the pure quadratic coefficient b_{ii}, the efficiency of SORD of second type using PBD is comparing with SORD of first type using PBD, here the scaling factor the equation (4.2) is applied. The relative efficiency SORD of second type using PBD versus SORD of first type using PBD is as follows based on the equation (5.1)

$$E\left(\frac{\text{second type of PBD}}{\text{first type of PBD}}\right)=\frac{\sigma^2 e_1\left(\frac{r2^{t(k)}+2a^2}{b2^{t(k)}+2v+m_0}\right)^2\left(b2^{t(k)}+2v+m_0\right)}{\sigma^2 e_2\left(\frac{r2^{t(k)}+2a_1^2+2a_2^2}{b2^{t(k)}+4v+n_0}\right)^2\left(b2^{t(k)}+4v+n_0\right)}$$

$$=\frac{e_1\left(r2^{t(k)}+2a^2\right)^2\left(b2^{t(k)}+4v+n_0\right)}{e_2\left(r2^{t(k)}+2a_1^2+2a_2^2\right)^2\left(b2^{t(k)}+2v+m_0\right)} \quad (5.8)$$

Where,

$e_1=v(b_{ii})$ in SORD of first type using PBD and $e_2=v(b_{ii})$ in SORD of second type using PBD.

For example, in SORD of second type using PBD of design $(v=9,b=311,r=5,k_1=5,k_2=4,k_3=3,\lambda=2)$, $n_o=1$, $a_1=1$, $a_2=1.6266$, $e_2=0.0156$ and in SORD of first type using PBD $m_0=1$, $a=1.6818$, $e_1=0.0174$, lets us compare the relative efficiency of SORD of second type using PBD versus SORD of first type using PBD, if you get the equation (5.8) as 1.1731, then we conclude that the SORD of second type using PBD is more efficient than SORD of first type using PBD

5.3 Comparison in terms estimating the first order coefficient b_i

It can be discussed in the same process of the $V(b_i)$ by multiplying the scaling factor then $V(b_i)=\frac{\sigma^2}{(r2^{t(k)}+2a_1^2+2a_2^2)}$ is to be multiplied by $1/g^2$ then we get, $V(b_i)=\frac{\sigma^2}{(r2^{t(k)}4v+n_0)}$ similarly the $V(b_i)$ is multiplied by scaling factor in SORD of first type using PBD the we obtained $V(b_i)=\frac{\sigma^2}{(r2^{t(k)}+2v+n_0)}$ so finally we compare the relative efficiency of $E\left(\frac{\text{SORD of second type using PBD}}{\text{SORD of first type using PBD}}\right)$ and it obtained 1, then the efficiency of SORD of second type PBD is more efficient than SORD of first type using PBD

6. Conclusion

In this paper, second order rotatable designs of SORD of second type using PBD in which the axial points are indicated by two numbers a_1 and a_2 and it is called as SORD of second type using PBD. The variance covariance of the estimated parameters are studied and we evaluated for the SORD of second type using PBD is most orthogonal for second order response surface designs and the results of the orthogonality are given in the table 4.1.

The comparison between the SORD of second type using PBD versus SORD of first type using PBD for different coefficients are studied then we conclude that the SORD of second type using PBD is more efficient than SORD of first type using PBD. It is convenient to use the practical situations and give the more efficiency when compared to SORD of first type using PBD.

References

Box, G.E.P. and Hunter, J.S. (1957), Multifactor experimental designs for exploring response surfaces, ***Annals of Mathematical Statistics,*** 28, 195-241.

Chiranjeevi, P., Benhur, K. J. and Victorbabu, B.Re. (2021), Second order rotatable designs of second type using central composite designs, ***Asian Journal of Probability and Statistics,*** (In Press)

Chiranjeevi, P. and Victorbabu, B.Re. (2021), Second order rotatable designs of second type using balanced incomplete block designs, ***paper submitted for the possible publication***.

Das, M.N. and Narasimham, V.L. (1962), Construction of rotatable designs through balanced incomplete block designs, ***Annals of Mathematical Statistics,*** 33, 1421-1439.

Das, M.N. and Giri, N.C. (1999), Design and Analysis of Experiments, ***New Age International (P) Limited,*** New Delhi-110 002, India.

Das, M.N., Parasad, R. and Manocha, V.P. (1999), Response surface designs, symmetrical and asymmetrical, rotatable and modified, ***Statistics and Applications,*** 1, 17-34

Draper, N.R. and Guttman, I. (1988), An index of rotatability, ***Technometrics,*** 30, 105-112.

Draper, N.R. and Pukelsheim, F. (1990), Another look at rotatability, ***Technometrics,*** 32, 195-202.

Kim, Hyuk Joo. (2002), Extended central composite designs with the axial points indicated by two numbers, ***The Korean Communications in Statistics,*** 9, 595-605.

Kim, Hyuk Joo. (2019), On modified slope rotatability of central composite designs with two axial values, ***The Korean Journal of Applied Statistics,*** 32, 867-878.

Kim, Hyuk Joo and Ko, Yun Mi. (2004), On slope rotatability of central composite designs of second type, ***The Korean Communications in Statistics,*** 11, 121-137.

Khuri, A.I. (1988), A measure of rotatability for response surface designs, ***Technometrics,*** 30, 95-104.

Myers, R.H. (1976), Response surface methodology, Blacksburg, VA: Author (distributed by Edwards Brothers, Ann Arbor, MI).

Park, S.H. Lim, J.H. and Baba, Y. (1993), A measure of rotatability for second order response surface designs, ***Annals of the Institute of Statistical Mathematics,*** 45, 655-664.

Park, H.J. and Park, S.H. (2010), Extension of central composite designs for second order response surface model building, ***Communications in Statistics-Theory and Methods,*** 39, 1202-1211.

Raghavarao, D. (1971), Constructions and combinatorial problems in Design of Experiments, ***John Wiley, New York***

Surekha, Ch.V.V.S. and Victorbabu, B. Re. (2012), Constriction of measure of rotatable central composite designs, ***International Journal of Agricultural and Statistical Sciences,*** 8, 1-6.

Tyagi, H.N. (1964), On the construction of second order and third order rotatable designs through pairwise balanced designs and doubly balanced designs, ***Calcutta Statistical Association Bulletin,*** 13, 150-162.

Victorbabu, B.Re. (2007), On second order rotatable designs – a review, ***International Journal of Agricultural and Statistical Sciences,*** 3, 201-209.

Victorbabu, B.Re. (2006), Construction of modified second order rotatable and second order slope rotatable designs using a pair of balanced incomplete block designs, ***Sri Lankan Journal of Applied Statistics,*** 7, 39-53.

Victorbabu, B.Re. (2009), Construction of modified second order rotatable designs using a pair of symmetrical unequal block arrangements with two unequal block sizes, ***Acharya Nagarjuna University: Journal of Physical Sciences*** 1, 73-80.

Victorbabu, B.Re. and Vasundharadevi, V. (2005) Modified second order response surface designs, rotatable designs using BIBD, ***Sri Lankan Journal of Applied Statistics,*** 6, 1-11.

Victorbabu, B.Re., Vasundharadevi, V. and Viswanadam, B. (2006), Modified second order response surface designs, using pairwise block designs, ***Advances and Applications in Statistics,*** 6, 323-334.

Victorbabu, B.Re., Vasundharadevi, V. and Viswanadam, B. (2008), Modified second order response surface designs, using central composite designs, ***Canadian Journal of Pure and Applied Sciences,*** 2, 289-294.

Victorbabu, B.Re. and Vasundharadevi, V. (2008), Modified second order response surface designs, rotatable designs using symmetrical unequal block arrangements with two unequal block sizes, ***Pakistan Journal of Statistics,*** 24, 67-76.

Victorbabu, B.Re. and Surekha, Ch.V.V.S. (2013), A note on measure of rotatability for second order response surface designs using incomplete block designs, ***Journal of Statistics: Advances in Theory and Applications,*** 1, 137-151.

Victorbabu, B.Re. and Surekha, Ch.V.V.S. (2015), A note on a measure of rotatability for second order response surface designs using balanced incomplete block designs, ***Thailand Statistician,*** 13, 97-110.