Original Research Article

Computational Analysis of the Stability of 2D Heat Equation on Elliptical Domain Using FDM

ABSTRACT

Aims: The aim and objective of the study to derive and analyze the stability of the finite difference schemes in relation to the irregularity of domain.

Study design: First of all an elliptical domain has been constructed with the governing 2D heat equation that is discretized using the finite difference method. Then the stability condition has been defined and the numerical solution by writing MATLAB codes has been obtained with the stable values of time domain..

Place and Duration of Study: The work has been jointly conducted at the MUET, Jamshoro and QUEST, Nawabshah Pakistan from January 2019 to December 2019.

Methodology: The stability condition over an elliptical domain with the non-uniform step size depending upon the boundary tracing function is derived by using Von Neumann method.

Results: From the results it was revealed that stability region for the small number of mesh points remains larger and gets smaller as the number of mesh nodes is increased. Moreover, the ranges for the time steps are defined for varied spatial step sizes that help to find the stable solution.

Conclusion: The corresponding stability range for number of nodes N=10, 20, 30, 40, 50, and 60 was found respectively. Within this range the solution remains smooth as time increases. The results of this study attempt to provide the stable solution of partial differential equations on irregular domains.

Keywords: *Modeling and simulation, stability analysis, computational analysis, finite difference method, elliptical domain, heat equation.*

1. INTRODUCTION

The Partial Differential Equations (PDEs) are widely used in many fields of science and engineering and considered as the principal sources of providing the mathematical models to govern the physical situations [1]. The 2D heat equation is a parabolic partial differential equation which is widely used in many scientific and engineering problems for the purpose of simulating the time dependent diffusion of heat or energy in the physical domains. For a simple one dimensional case it is represented mathematically as follows:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where u is the dependent variable and c is the thermal diffusivity constant. Solution of heat equation is computed by variety methods including analytical and numerical methods. But when the heat equation is considered for 2-dimensional and 3-dimensional problems then the analytical solution becomes difficult or impossible in some cases. Then the numerical methods are best choice to solve the problem in 2D and 3D. The errors in the numerical methods are akin to the convergence behavior of the solution algorithm that may accumulate abruptly if the proper values of the time steps or mesh spacing are not selected. The consequences of such errors lead to the instability in the numerical solution and the situation becomes worst when the problem is defined over irregular domain. Without having any prior knowledge it becomes difficult to give the guarantee of convergence. For the convergence of solution by a numerical finite difference scheme the consistency and stability are the necessary and sufficient requirements respectively. The stability of finite difference numerical schemes can be investigated by a procedure known as Von Neumann stability analysis or Fourier method [3]. In the following section some related works have been reviewed that discuss the solution of heat equation by different methods and the stability of numerical schemes used.

Since this study is concerned with the solution of heat equation over irregular boundaries therefore the attempt is made to highlight previous works that have been used for irregular boundaries. In this regards Han et al. [4] have investigated a finite difference scheme for solving the variable coefficient Poisson and heat equations on irregular domains with Dirichlet boundary conditions. They considered non-graded Cartesian grids (grids for which the difference in size between two adjacent cells is not fixed) and employed a second order implicit discretization in time. A parallel solution approach for 2D heat equation was presented by [6] and they showed that the good numerical approximations can be obtained using finite difference method. A systematic and practical overview of the numerical solution of 1D heat equation using finite difference method was given by Gerald [7]. He used the MATLAB codes to find the differences between explicit finite time, centered space (FTCS) and implicit backward time, centered space (BTCS) and implicit Crank-Nicolson methods. The semi discretized heat equations over irregular domains were solved by Kazufumi et. al [8]. They used second and fourth order grid based finite difference methods derived from multivariable Taylor series expansion and included the idea of eigenvalues. Their methods offer systematic treatment of the general boundary conditions in two and three dimensions. [9] Proposed a new method to solve the steady state heat equation in 2D on irregular domains. They applied the method on two different types of meshes viz. irregular and semi irregular and concluded that their method can be efficiently used for solving PDEs over irregular domains. A mesh free method was used by [10] for solving 3D heat equation by explicit scheme and the stability of the scheme was addresses by taking irregularity of the points in account. Their results showed the improvement in the accuracy of the solution.

The first relationship between stability and convergence was hinted at by Courant, Friedrichs and Lewy (hence known as CFL condition) in the 1920's [12-14]. Then it was clearly identified Von Neumann in the 1940 [15]. Later it was brought into organized form by Lax and Richtmyer in the 1950s by stating a fundamental theorem Lax Equivalence Theorem [16]. However, in most of the cases Von Neumann stability analysis which is based on the Fourier series briefly described by [17] is applied.

In literature, a number of studies can be found where the Von Neumann stability analysis is applied to devise or analyze the well posedness of the problems of interest. In this regards [18- 27] have done extensive work to either utilize, modify or establish stability conditions akin to Von Neumann stability analysis. A comprehensive review of the recent and past technique can be found in [28]. However, the stability analysis of heat equation becomes difficult when the domain under consideration has nonlinear boundaries leading to irregular mesh spacing.

A significant work has been done to investigate the stability of 1D, 2D and 3D heat equation for different finite difference schemes ranging from explicit to implicit methods. However, the stability of heat equation can be difficult when it is applied on nonlinear boundaries or domains. This issue provides motivation for research in the present state of the art by applying 2D heat equation over nonlinear domain specifically over elliptical domain.

2. METHODOLOGY

In this study a 2D elliptic domain Ω with boundary $\partial\Omega$ is considered by using general equation of ellipse. Suppose that the domain is made of some thermally conductive material with diffusion coefficient c^2 and heated in some way by applying the initial and boundary conditions. Mathematically, this problem is governed by 2D heat equation as given by Eq. (3) below:

$$\frac{\partial u(x,y,t)}{\partial t} = c^2\left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right), \tag{3}$$

where $u(x,y,t)$ represents the temperature at any point $P(x,y)$ at specific time t. As particular case the temperature at the boundary of the ellipse is set as $u=100$ initially at time t=0 the temperature is $u=0$ on other than the boundary nodes (as shown in Figure 1). The governing 2D heat equation (3) is discretized by using the explicit forward Euler and centered finite difference schemes for time and space parameters respectively [29].

$$u_{j,k,n+1} = u_{j,k,n} + \Delta t.c^2\left(\begin{array}{l}\dfrac{1}{(\Delta x)^2}\left(u_{j+1,k,n} - 2u_{j,k,n} + u_{j-1,k,n}\right) + \\ \dfrac{1}{\Delta y_1 \Delta y_2\left(\Delta y_1 + \Delta y_2\right)}\left(2\Delta y_1 u_{j,k+1,n} - 2\left(\Delta y_1 + \Delta y_2\right)u_{j,k,n} + 2\Delta y_2 u_{j,k-1,n}\right)\end{array}\right), \tag{4}$$

where $\Delta y_1 = y_i - y_{i-1}$ and $\Delta y_2 = y_{i+1} - y_i$. Equation (4) finds the numerical solution on each interior node *(i, j)* at the time *(n+1)* th time step based on the solution of previous n^{th} time step. Figure 2 shows the finite difference mesh of the discretized domain with N=50 cells along x-axis and N=50 cells along y-axis. In order to reduce the computational cost the exterior cells are removed and solution be computed only on the interior nodes (see Figure 3). The computational analysis of the mesh elements and mesh nodes for different choices of N is given in Table 1.

Table 1: Analysis of the finite difference mesh parameters

S. No	N	Δx	Total Nodes TN	Number of exterior nodes, EN	Number of boundary nodes, BN	Number of interior nodes, IN	Number of cells, CN	Number of exterior cells EC	Number of interior cells IC	Number of common cells CC
1	10	0.60000000	121	60	22	39	100	40	60	22
2	20	0.30000000	441	220	42	179	400	180	220	42
3	30	0.20000000	961	480	62	419	900	420	480	62
4	40	0.15000000	1681	840	82	759	1600	760	840	82
5	50	0.12000000	2601	1300	102	1199	2500	1200	1300	102
6	60	0.10000000	3721	1860	122	1739	3600	1740	1860	122

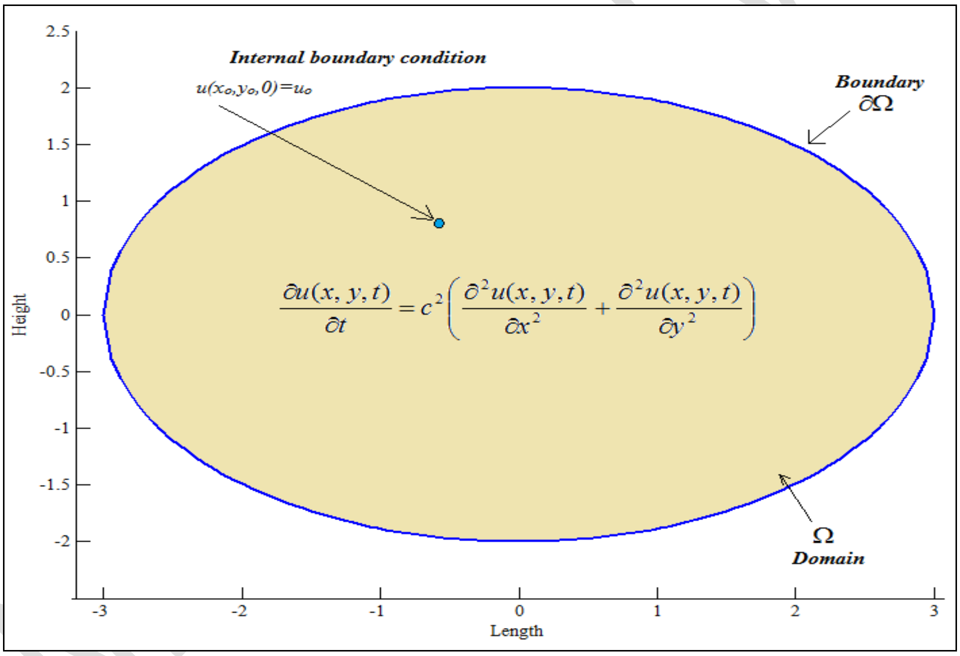

Figure 1.Schematic of elliptic domain with applied heat equation and boundary conditions.

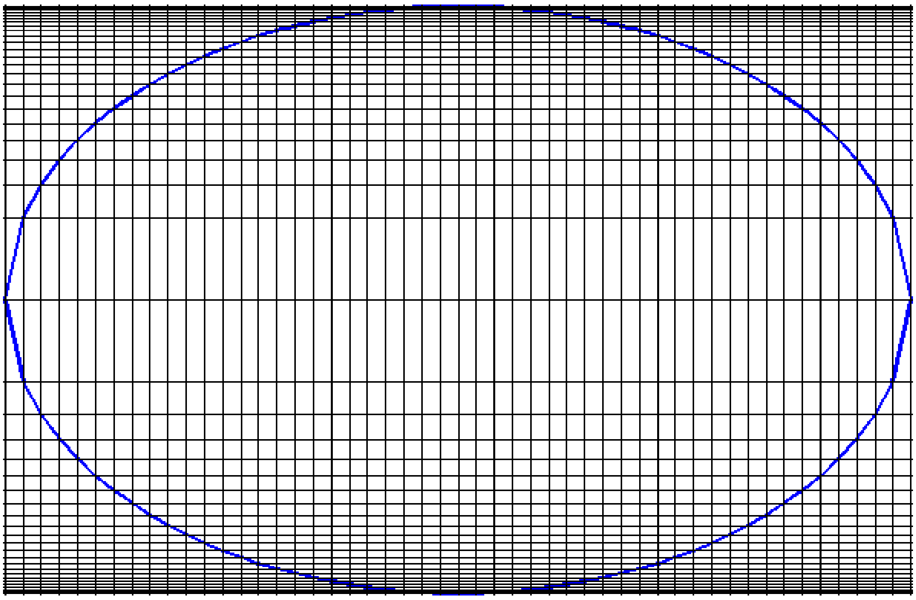

Figure 2. Schematic of discretized domain for 50 x 50 mesh

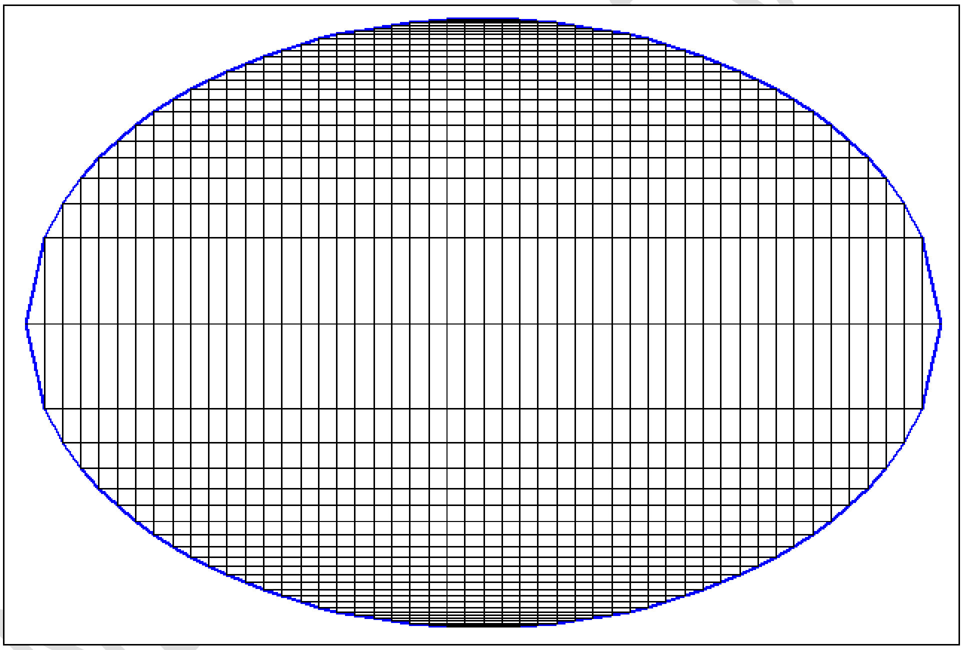

Figure 3. Simplified mesh of the domain, where extra cells are removed

With the aim of finding the stability condition for the heat equation with unequal mesh spacing the Von-Neumann stability method [30] is redefined by taking the average of Δy_1 and Δy_2. Thus by substituting $u_{j,k,n} = e^{il\Delta x(j)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)}$ in Eq. (4) yields the new equation as

$$u_{j,k,n+1} = e^{il\Delta x(j)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)} + \frac{\Delta t}{(\Delta x)^2}c^2\left[e^{il\Delta x(j+1)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)} - 2e^{il\Delta x(j)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)} + e^{il\Delta x(j-1)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)}\right] + \frac{\Delta t}{\Delta y_1\Delta y_2\left(\Delta y_1+\Delta y_2\right)}c^2\left[2\Delta y_1 e^{il\Delta x(j)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k+1)} - 2\left(\Delta y_1+\Delta y_2\right)e^{il\Delta x(j)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)} + 2\Delta y_2 e^{il\Delta x(j)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k-1)}\right], \quad (5)$$

by taking $e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)}$ common and re-arranging the terms the following Eq. (8) is obtained,

$$u_{j,k,n+1}=e^{il\Delta x(j)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)}\left[\begin{array}{l}1+\frac{\Delta t}{(\Delta x)^2}c^2\left(e^{il\Delta x}-2+e^{-il\Delta x}\right)+\\ \frac{\Delta t}{\Delta y_1\Delta y_2\left(\Delta y_1+\Delta y_2\right)}c^2\left(2\Delta y_1 e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)}-2\left(\Delta y_1+\Delta y_2\right)+2\Delta y_2 e^{-im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)}\right)\end{array}\right], \tag{6}$$

Now assume that the expression in the square brackets is a function of k^{th} time step and is denoted by *G(k)* as given in Eq. (7)

$$G(k)=1-2\frac{\Delta t}{(\Delta x)^2}c^2\left[1-\cos(l\Delta x)\right]+\frac{\Delta t}{\Delta y_1\Delta y_2\left(\Delta y_1+\Delta y_2\right)}c^2\left(2\Delta y_1 e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)}+2\Delta y_2 e^{-im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)}-2\left(\Delta y_1+\Delta y_2\right)\right) \tag{7}$$

Then expanding exponential Euler formula the Eq. (7) takes the following form,

$$G(k)=1-2\frac{\Delta t}{(\Delta x)^2}c^2\left[1-\cos(l\Delta x)\right]+\frac{\Delta t}{\Delta y_1\Delta y_2\left(\Delta y_1+\Delta y_2\right)}c^2\left[\begin{array}{l}2\left(\Delta y_1\cos\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)+\imath\sin\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)\right)\\ +\Delta y_2\left(\cos\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)+\imath\sin\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)\right)\\ -2\left(\Delta y_1+\Delta y_2\right)\end{array}\right], \tag{8}$$

or can be expressesd as follows,

$$G(k)=1-2\frac{\Delta t}{(\Delta x)^2}c^2\left[1-\cos(l\Delta x)\right]+\frac{\Delta t}{\Delta y_1\Delta y_2\left(\Delta y_1+\Delta y_2\right)}c^2\left[2\left(\begin{array}{l}\cos\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)\left(\Delta y_1+\Delta y_2\right)\\ +\imath\sin\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)\left(\Delta y_1-\Delta y_2\right)\\ -2\left(\Delta y_1+\Delta y_2\right)\end{array}\right)\right], \tag{9}$$

$$G(k)=1-2\frac{\Delta t}{(\Delta x)^2}c^2\left[1-\cos(l\Delta x)\right]+\frac{\Delta t}{\Delta y_1\Delta y_2\left(\Delta y_1+\Delta y_2\right)}c^2\left[\begin{array}{l}2\left(\Delta y_1+\Delta y_2\right)\left(\cos\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)-1\right)+\\ 2\imath\sin\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)\left(\Delta y_1-\Delta y_2\right)\end{array}\right], \tag{10}$$

$$\begin{aligned}G(k)=1-2\frac{\Delta t}{(\Delta x)^2}c^2\left[1-\cos(l\Delta x)\right]-2\frac{\Delta t}{\Delta y_1\Delta y_2}c^2\left[1-\cos\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)\right]+\\ \frac{\Delta t}{\Delta y_1\Delta y_2\left(\Delta y_1+\Delta y_2\right)}c^2\left[2\imath\sin\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)\left(\Delta y_1-\Delta y_2\right)\right].\end{aligned} \tag{11}$$

Consider if worst case $l\Delta x=\pi=m\left(\frac{\Delta y_1+\Delta y_2}{2}\right)$, then $G(k)=1-4\frac{\Delta t}{(\Delta x)^2}c^2-4\frac{\Delta t}{\Delta y_1\Delta y_2}c^2$, or

$$G(k)=1-4\left[\frac{\Delta t}{(\Delta x)^2}c^2+\frac{\Delta t}{\Delta y_1\Delta y_2}c^2\right].$$

Then by definition of stability $|G|\le 1,\ \forall l,m$ which yields the stability condition according to the Von-Neumann stability condition, finally the result is obtained as follows:

$$\frac{\Delta t}{(\Delta x)^2}c^2+\frac{\Delta t}{\Delta y_1\Delta y_2}c^2\le\frac{1}{2}\Leftrightarrow\Delta t\le\frac{1}{2c^2}\left[\frac{1}{(\Delta x)^2}+\frac{1}{\Delta y_1\Delta y_2}\right]^{-1}, \quad (12)$$

which is the required stability condition for the numerical solution of Eq. (3).

3. RESULTS AND DISCUSSION

The numerical solution for the defined problem is computed with the explicit finite difference scheme with the stability condition (12) by writing a user defined code on MATLAB. First of all for the six different choices of *N* the different meshes have been generated and the stable ranges of the time increment Δt have been found.

The following **Table 2** shows the variation in the stability range in relation to different step sizes and functional increments. The interval of stability range for time step gets much smaller as the number of mesh nodes increases. The same behavior of stabile time step in relation to the varied increment Δy_1 along y-axis has shown in Figure 4. A more clear 3D representation of the Δt depending upon Δy_1 and Δy_2 has exhibited by Figure 3 which reveals that the stability region for variable spatial increments scales down as the number of mesh points are increased. In order to validate the smoothness of temperature diffusion the simulation profiles from the numerical solution have been obtained and shown in the Figure 6 (a) through Figure (j). The simulation profiles are taken for t=0 to t=1; and then the solution is interpolated for t=0.1 , t=0.2, …,t=1.0. From the figures it can be bee seen that the effect of the heat diffuses from boundary to interior region as the time increases. If the time is let to further increase the time dependent diffusion will lead to the stationary behavior.

Table 2: Analysis of the Stable range of the time step

S. No	N	Δx	min(Δy)	max(Δy)	mean(Δy)	min(Δt)	max(Δt)	Stable range of Δt
1	10	0.60000000	0.04040800	1.20000000	0.62020410	0.00252124	0.14400000	[0.00252124, 0.144]
2	20	0.30000000	0.01002500	0.87177900	0.44090000	0.00015000	0.04023500	[0.00015, 0.040235]
3	30	0.20000000	0.00444930	0.71802100	0.36123500	0.00002978	0.01856000	[2.98E-5, 0.01856]
4	40	0.15000000	0.00250150	0.62449900	0.31350000	0.00000940	0.01063630	[9.40E-6, 0.0106363]
5	50	0.12000000	0.00160000	0.56000000	0.28080000	0.00000385	0.00688300	[3.85E-6, 0.006883]
6	60	0.10000000	0.00111142	0.51207638	0.25659390	0.00000185	0.00481633	[1.8542E-06, 0.0048]

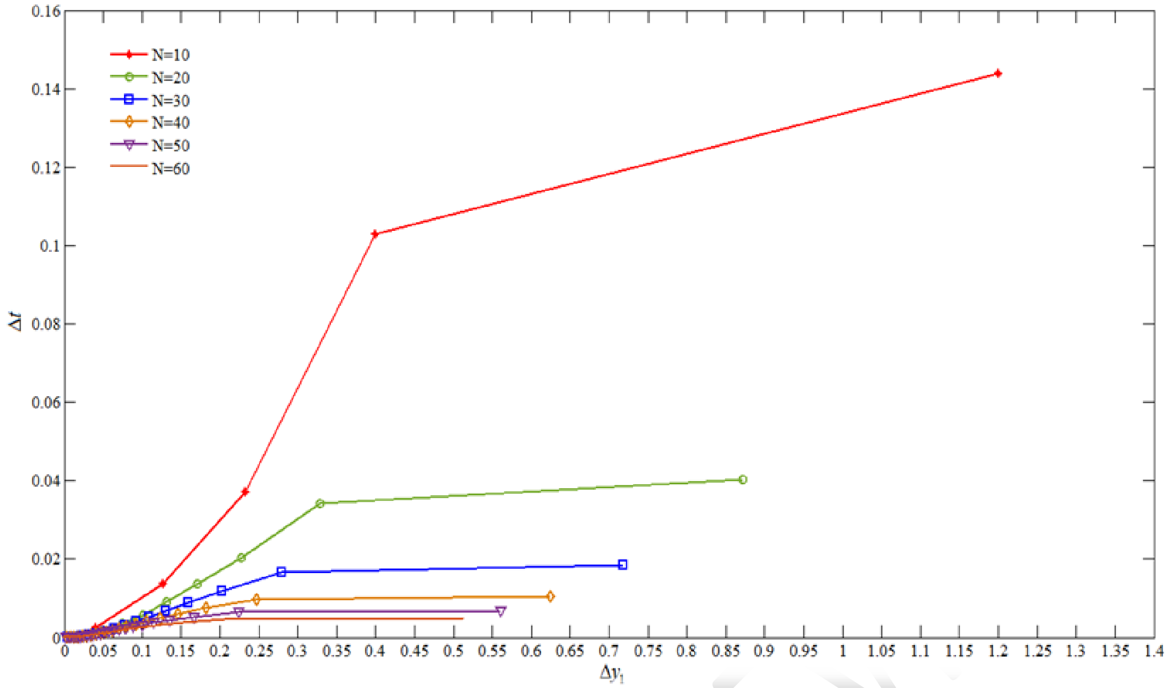

Figure 4. Relation between functional increment and the time step at different mesh size.

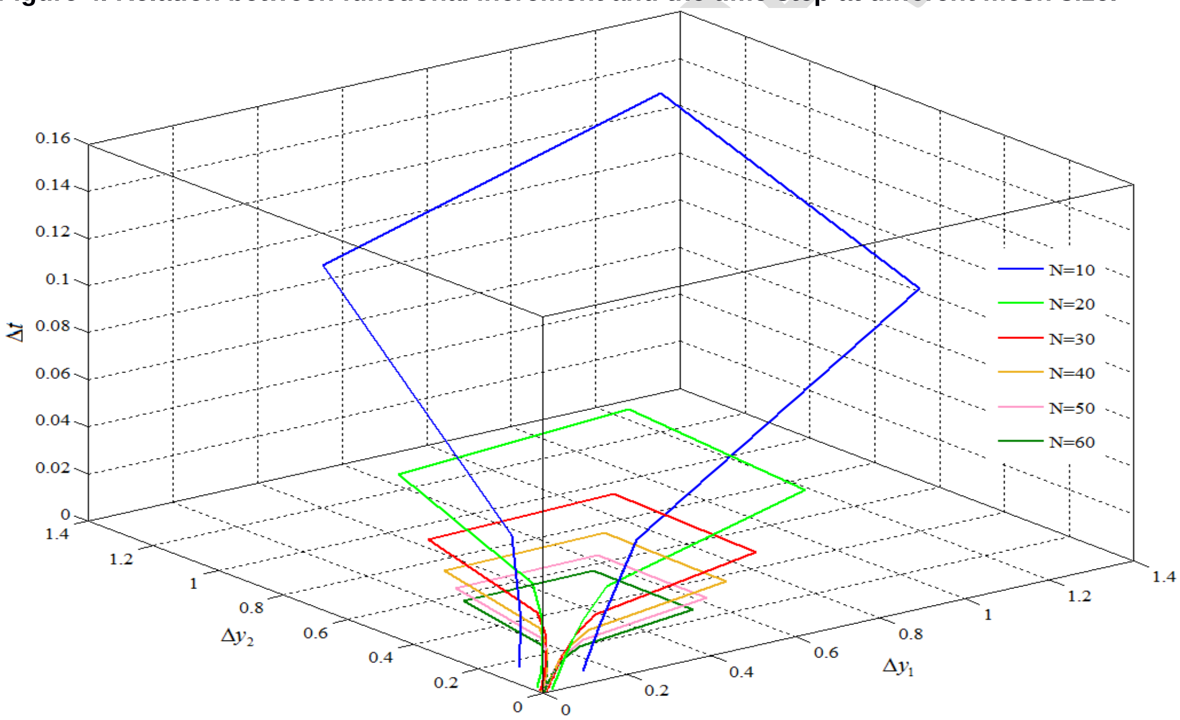

Figure 5. The stability region for time step at different mesh size with respect to functional increments

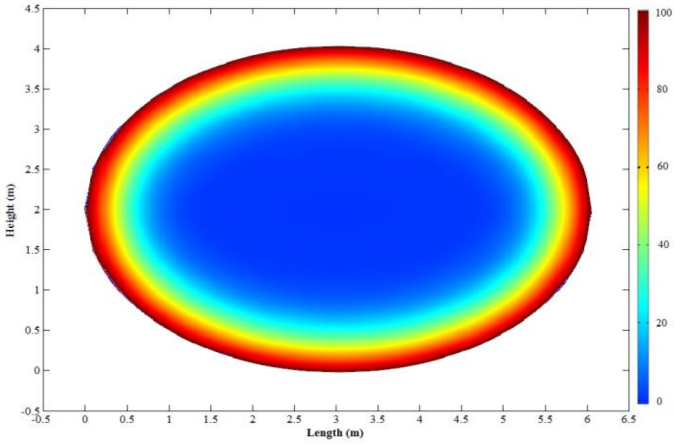

Figure 6 (a). Simulation of temperature diffusion in the domain at t=0.1

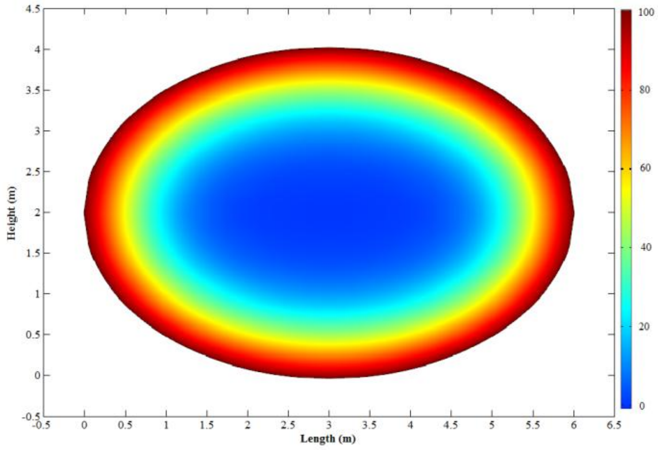

Figure 6 (b). Simulation of temperature diffusion in the domain at t=0.2.

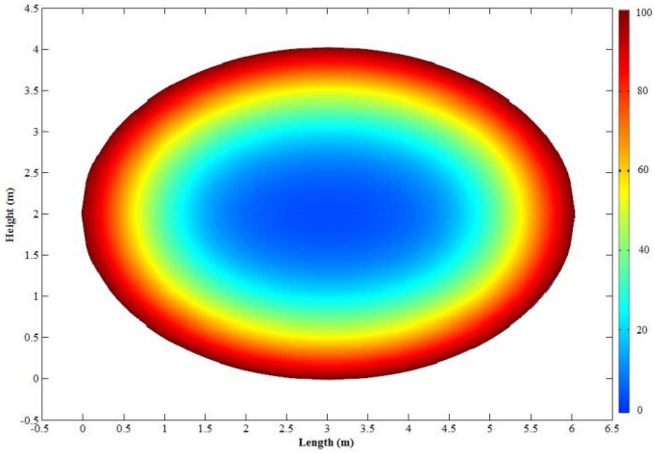

Figure 6 (c). Simulation of temperature diffusion in the domain at t=0.3

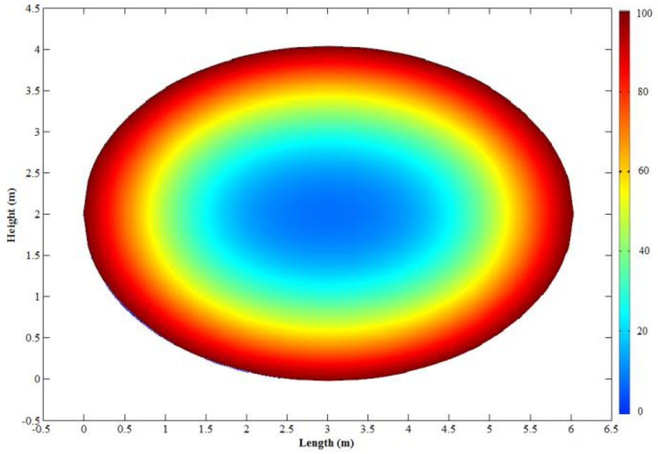

Figure 6 (d). Simulation of temperature diffusion in the domain at t=0.4

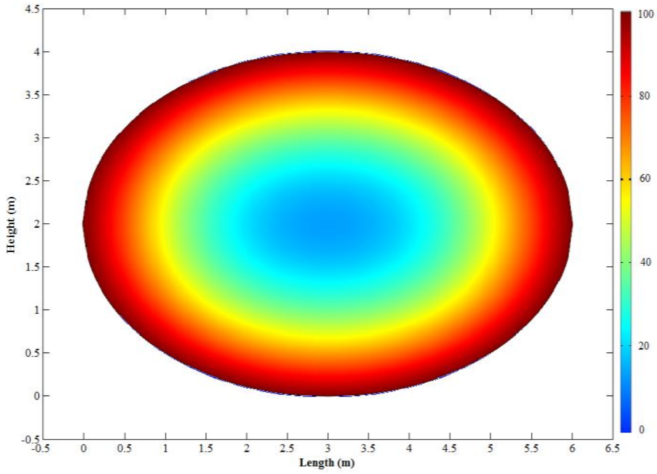

Figure 6 (e). Simulation of temperature diffusion in the domain at t=0.5

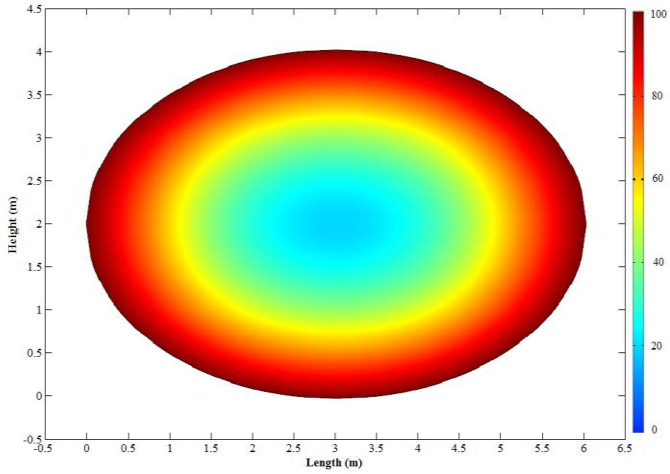

Figure 6 (f). Simulation of temperature diffusion in the domain at t=0.6

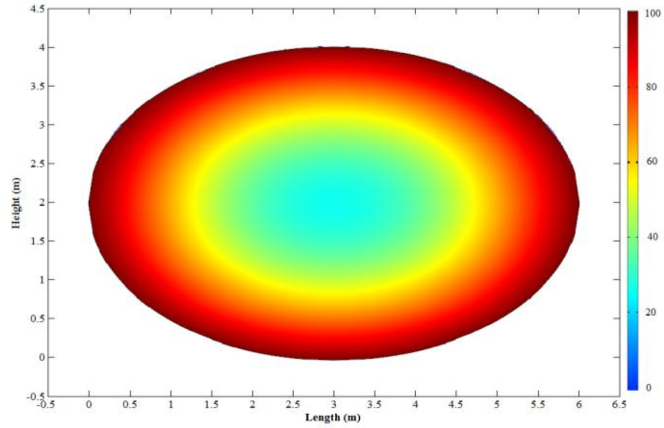

Figure 6 (g). Simulation of temperature diffusion in the domain at t=0.7

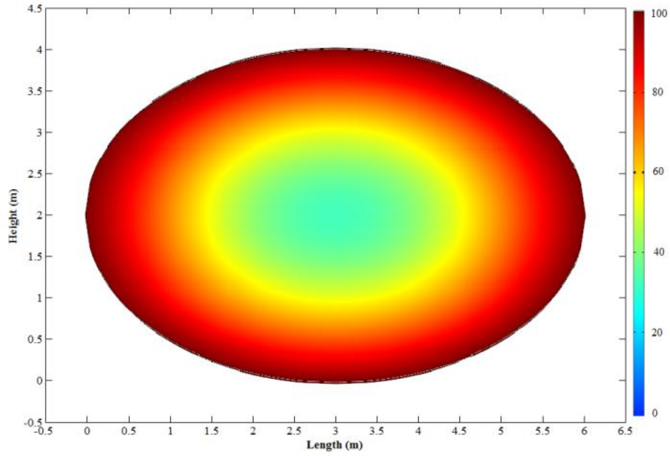

Figure 6 (h). Simulation of temperature diffusion in the domain at t=0.8.

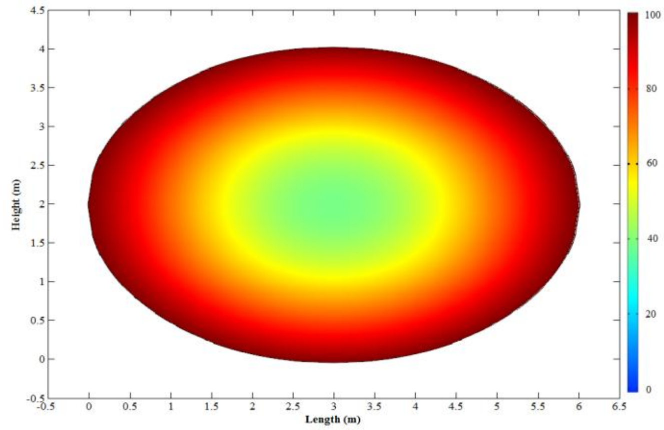

Figure 6 (i). Simulation of temperature diffusion in the domain at t=0.9

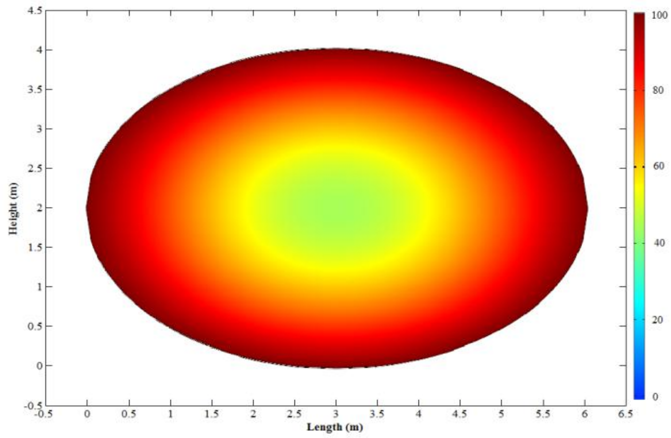

Figure 6 (j). Simulation of temperature diffusion in the domain at t=1.0

4. CONCLUSION

In this study the stability analysis of the finite difference solution of 2D heat equation was investigated. The main purpose was to find out the stability criteria for the explicit finite difference scheme on irregular domain. Where the domain boundary was constructed by using the equation of ellipse. The problem of stability occurs when the mesh size is unequal along the y-axis due to functional increments. Therefore, the finite difference scheme was redefined for unequal step size along y-axis, the analogous Von-Neumann stability analysis was worked out and the general formula for such problem was obtained. From the results it was revealed that stability region for the small number of mesh points remains larger and then stability region gets smaller as the number of nodes are increased. The corresponding stability range for N=10, 20, 30, 40, 50, and 60 was found respectively. Within that range the solution remains smooth as time increases. The results of this study attempt to provide the stable and accurate solution of partial differential equations on irregular domains. The similar work can be done for other types of PDEs such as hyperbolic, elliptical, etc; and the methodology can be extended to 3D.

REFERENCES

[1] https://en.wikipedia.org/wiki/partial_differential_equation, retrieved in May 2016.
[2] Olsen-Kettle, L. (2011). Numerical solution of partial differential equations.
[3] E. Isaacson, & H.B. Keller, Analysis of Numerical Methods, 1994, 541.
[4] Han Chen, Cho Hong Min, Frederick Gibou, A Supra-Convergent Finite Difference Scheme for the Poisson and Heat Equations on Irregular Domains and Non-Graded Adaptive Grids, 2006.
[5] Peter McCorquodale, Phillip Colella, and Hans Johansen, A Cartesian Grid Embedded Boundary Method for the Heat Equation on Irregular Domains, doi:10.1006/jcph.2001.6900, available online at http://www.idealibrary.com.
[6] Verena Horak & Peter Gruber, Parallel Numerical Solution of 2-D Heat Equation, Parallel Numerics ’05, 47-56.
[7] Gerald W. Recktenwald, Finite-Difference Approximations to the Heat Equation, 2011.
[8] Kazufumi Ito, Zhilin Li and Yaw Kyei, Higher-Order, Cartesian Grid Based Finite Difference Schemes for Elliptic Equations on Irregular Domains, SIAM Journal on Scientific Computing Volume 27, Issue 1, 2005 pages 346-367.
[9] J. Izadian and N. Karamooz, New Method for Solving Poisson Equation on Irregular Domains, Applied Mathematical Sciences, Vol. 6, 2012, no. 8, 369 – 380.
[10] L.Gavete, J.J. Benito and F. Urena, Generalized finite differences for solving 3D elliptic and parabolic equations, Applied Mathematical Modelling, Volume 40, Issue 2, 15 January 2016, Pages 955-965.
[11] Gilberto E. Urroz, Convergence, Stability, and Consistency of Finite Difference Schemes in the Solution of Partial Differential Equations, July 2004.
[12] Courant, R.; Friedrichs, K.; Lewy, H. (1928), "Über die partiellen Differenzengleichungen der mathematischen Physik", Mathematische Annalen (in German), 100 (1): 32-74.
[13] Courant, R.; Friedrichs, K.; Lewy, H. (September 1956), On the partial difference equations of mathematical physics, AEC Research and Development Report, NYO-7689, New York: AEC Computing and Applied Mathematics.
[14] Courant, R.; Friedrichs, K.; Lewy, H. (March 1967) [1928], "On the partial difference equations of mathematical physics", IBM Journal of Research and Development, 11 (2): 215-34.
[15] Charney, J. G.; Fjørtoft, R.; von Neumann, J. (1950), "Numerical Integration of the Barotropic Vorticity Equation", Tellus, 2: 237–254, doi:10.1111/j.2153-3490.1950.tb00336.x.
[16] Lax, P. D.; Richtmyer, R. D. Survey of the stability of linear finite difference equations. Comm. Pure Appl. Math. 9 (1956), 267--293 MR 79204, doi:10.1002/cpa.3160090206.
[17] Crank, J.; Nicolson, P. (1947), "A Practical Method for Numerical Evaluation of Solutions of Partial Differential Equations of Heat Conduction Type", Proc. Camb. Phil. Soc., 43: 50–67,doi:10.1007/BF02127704.
[18] S.B. Yuste, Weighted average finite difference methods for fractional diffusion equations, Journal of Computational Physics 216 (2006) 264–274.
[19] S. B. Yuste and L. Acedo, An Explicit Finite Difference Method and a New von Neumann-Type Stability Analysis for Fractional Diffusion Equations, SIAM Journal on Numerical Analysis, 2005, Vol. 42, No. 5: pp. 1862-1874.
[20] S. Bilbao, "Modeling of Complex Geometries and Boundary Conditions in Finite Difference/Finite Volume Time Domain Room Acoustics Simulation," in *IEEE Transactions on Audio, Speech, and Language Processing*, vol. 21, no. 7, pp. 1524-1533, July 2013.
[21] Jincheng Ren, Zhi-zhong Sun, Xuan Zhao, Compact difference scheme for the fractional sub-diffusion equation with Neumann boundary conditions, Journal of Computational Physics, Volume 232, Issue 1, 1 January 2013, Pages 456-467, ISSN 0021-9991, http://dx.doi.org/10.1016/j.jcp.2012.08.026.
[22] Ercília Sousa, Can Li, A weighted finite difference method for the fractional diffusion equation based on the Riemann–Liouville derivative, Applied Numerical Mathematics, Volume 90, April 2015, Pages 22-37, ISSN 0168-9274, http://dx.doi.org/10.1016/j.apnum.2014.11.007.
[23] Ercília Sousa, A second order explicit finite difference method for the fractional advection diffusion equation, Computers & Mathematics with Applications, Volume 64, Issue 10, November 2012, Pages 3141-3152, ISSN 0898-1221, http://dx.doi.org/10.1016/j.camwa.2012.03.002.
[24] N. H. Sweilam, M. M. Khader, A. M. S. Mahdy, Crank-Nicolson finite difference method for solving time-fractional diffusion equation, Journal of Fractional Calculus and Applications, Vol. 2. Jan. 2012, No. 2, pp. 1-9.
[25] J. Quintana-Murillo, S. B. Yuste, A finite difference method with non-uniform timesteps for fractional diffusion and diffusion-wave equations, The European Physical Journal Special Topics, September 2013, Volume 222, Issue 8, pp 1987–1998.

[26] Norihiro Watanabe and Olaf Kolditz, Numerical stability analysis of two-dimensional solute transport along a discrete fracture in a porous rock matrix, Water Resources Research, Volume 51, Issue 7, pages 5855–5868, July 2015.
[27] Jiequan Li, Zhicheng Yang, The von Neumann analysis and modified equation approach for finite difference schemes, Applied Mathematics and Computation, Volume 225, 1 December 2013, Pages 610-621, ISSN 0096-3003, http://dx.doi.org/10.1016/j.amc.2013.09.046.
[28] W. Ehlers, S. Zinatbakhsh and B. Markert, Stability analysis of finite difference schemes revisited: A study of decoupled solution strategies for coupled multifield problems, Int. J. Numer. Meth. Engng 2013; 94:758–78.
[29] D. Seibold, Von Neumann Stability Analysis - MIT OpenCourseWare, 2005. ocw.mit.edu/courses/mathematics/18...equations.../MIT18_336S09_lec14.pdf
[30] Singh, A. K., & Bhadauria, B. S. (2009). Finite difference formulae for unequal sub-intervals using Lagrange's interpolation formula. *Int. J. Math. Anal*, *3*(17), 815.
[31] D. Seibold, Von Neumann Stability Analysis - MIT OpenCourseWare, 2005. ocw.mit.edu/ courses/mathematics/18...equations.../MIT18_336S09_lec14.pdf.