Comparison of Mixed and Multiplicative Models when Trend Cycle Component is Linear

Abstract: The purpose of this study is to present the linear trend cycle component with the emphasis on the choice between mixed and multiplication models in time series analysis. Most of the existing studies have adequately dwelt more on choice of model between additive and multiplicative, with little or no regards to the mixed model. The main aim of this study is to compare the row, column and overall means and variances for mixed and multiplicative models using Buys-Ballot table for seasonal time series. Specific objectives are 1) to obtain and compare the expected values of means for mixed and multiplicative models 2) to estimate and compare trend parameters and seasonal indices (when there is no trend, that is (b = 0)). The study indicate that column variances ($\hat{\sigma}_j^2$) of the Buys-Ballot table depends on the season j only through the square of the seasonal effect S_j^2 for mixed model and it is for multiplicative model, a quadratic function of the column j and square of the seasonal effect S_j^2.

Keywords: Time Series Decomposition, Trend Cycle Component, Mixed Model, Multiplicative Model, Expected Value, Buys-Ballot Table,

UNDER PEER REVIEW

1 Introduction

Time series decomposition method involve the separation of an observed time series into components representing trend (long term direction), seasonal (calendar related movements), cyclical (long term oscillations) and irregular (short term fluctuations) components. For short period of time series data, the cyclical component is superimposed into the trend and the observed time series $\left(X_t,\ t = 1, 2, ..., n\right)$ can be decomposed into the trend-cycle component $\left(M_t\right)$, seasonal component $\left(S_t\right)$ and the irregular/residual component $\left(e_t\right)$, Chatfield; 2004 [1]. Therefore, the decomposition models are

Additive Model:

$$X_t = M_t + S_t + e_t \qquad (1)$$

Multiplicative Model:

$$X_t = M_t \times S_t \times e_t \quad (2)$$

and Mixed Model

$$X_t = M_t \times S_t + e_t . \quad (3)$$

It is always assumed that the seasonal effect, when it exists, has period *s*, that is, it repeats after *s* time periods.

$$S_{t+s} = S_t \text{, for all t} \quad (4)$$

Additive model is assumed that, the sum of the seasonal components over a complete period/year is zero, ie ,

$$\sum_{j=1}^{s} S_{t+j} = 0 . \quad (5)$$

Similarly, the assumption for both multiplicative and mixed models is that, the sum of the seasonal components over a complete period is *s*.

$$\sum_{j=1}^{s} S_{t+j} = s . \quad (6)$$

The additive model, which is the simpler to use arithmetically, assumes that the actual time series data is the sum of the four basic separate effects. It assumes that the effect of the trend, the season, the cycles and the residuals are equal in absolute terms throughout the period of time. This assumption is usually true when short periods are involved or where the rate of growth or decline in the trend is small and transformation is not needed. However, this study will consider mixed and multiplicative models.

Oladugba, *et al*, [2] presented brief description between additive and multiplicative models in time series decomposition. In their opinion, the seasonal fluctuation exhibits constant amplitude with respect to the trend in additive model. While amplitude of the seasonal fluctuation depends on trend in multiplicative model. Nwogu, *et al* [3] and Dozie, *et al* [4] proposed Chi-Square test based on the seasonal variances of the Buys-Ballot table. The

UNDER PEER REVIEW

test has been theoretically verified to be quite successful and efficient for choice between mixed and multiplicative models in time series analysis.

2 Methodology.

The method adopted in this study is Buys-Ballot procedure for time series decomposition. For further details of Buys-Ballot table/procedure, see Wei [5], Iwueze and Nwogu [6, 7 and 8], Iwueze and Ohakwe [9], Dozie [10], Dozie, *et al,* [4], Dozie and Ijomah [11]. Nwogu, *et al,* [3] and Dozie, *et al,* [4] derived the row, column and grand means and variances of the Buys-Ballot table for mixed model given in Table 1, while comparing them with those of the multiplicative model. As Table 1 shows, the rows, columns and overall means and variances are not the same for both mixed and multiplicative models. However, the expected values of rows, columns and overall means are the same for both multiplicative and mixed models (see Table 2).

2.1 Expected values of means for mixed and multiplicative models.

Using the expression in Table 1, expected values of the means for mixed and multiplicative models are obtained. The row mean for mixed model is;

$$\bar{X}_i = a - bs(i-1) + \frac{b}{s}\sum_{j=1}^{s} jS_j + \bar{e}_i \tag{7}$$

$$E\left(\bar{X}_i\right) = E\left[a - bs(i-1) + \frac{b}{s}\sum_{j=1}^{s} jS_j\right] + E\left(\bar{e}_i\right)$$

Hence, the expected value of row mean is

$$E\left(\bar{X}_i\right) = a - bs + bsi + \frac{b}{s}\sum_{j=1}^{s} jS_j \tag{8}$$

where $E\left(\bar{e}_i\right) = 0$

For multiplicative model, the row mean is

UNDER PEER REVIEW

$$\bar{X}_i = \left[a - bs(i-1) + \frac{b}{s}\sum_{j=1}^{s} jS_j + bsi \right] \bar{e}_i \tag{9}$$

$$E\left(\bar{X}_i\right) = E\left[a - bs(i-1) + \frac{b}{s}\sum_{j=1}^{s} jS_j + bsi \right] E\left(\bar{e}_i\right)$$

Thus, the expected value of row mean is

$$E\left(\bar{X}_i\right) = a - bs + bsi + \frac{b}{2}\sum_{j=1}^{s} jS_j \tag{10}$$

where $E\left(\bar{e}_i\right) = 1$

The column mean for mixed model is

$$\bar{X}_{.j} = \left[a + b\left(\frac{n-s}{2}\right) + bj \right] S_j + \bar{e}_{.j} \tag{11}$$

$$E\left(\bar{X}_{.j}\right) = E\left[a + b\left(\frac{n-s}{2}\right) + bj \right] E\left(S_j\right) + E\left(\bar{e}_{.j}\right)$$

Therefore, the expected value of the column mean is

$$E\left(\bar{X}_{.j}\right) = \left[a + b\left(\frac{n-s}{2}\right) + bj \right] S_j \tag{12}$$

where $E\left(\bar{e}_{.j}\right) = 0$

for multiplicative model, the column mean is

$$\bar{X}_{.j} = \left[a\bar{e}_{.j} + \frac{bs}{m}\sum_{i=1}^{m} ie_{ij} - bs\bar{e}_{.j} + bj\bar{e}_{.j} \right] S_j \tag{13}$$

$$E\left(\bar{X}_{.j}\right) = E\left[a\bar{e}_{.j} + \frac{bs}{m}\sum_{i=1}^{m}(i-1) - bs + bj \right] E\left(S_j\right)$$

$$= \left[a + \frac{bsm}{m}\frac{(m-1)}{2} - bs + bj \right] S_j$$

Hence, the expected value of column mean is

UNDER PEER REVIEW

$$E\left(\bar{X}_{.j}\right) = \left[a + b\left(\frac{n-s}{2}\right) + bj\right]S_j \tag{14}$$

Overall mean for mixed model is

$$\bar{X}_{..} = a + b\left(\frac{n-s}{2}\right) + bc_1 + \bar{e}_{..} \tag{15}$$

$$E\left(\bar{X}_{..}\right) = E\left[a + b\left(\frac{n-s}{2}\right) + bc_1\right] + E\left(\bar{e}_{..}\right)$$

Hence , the expected value for the overall mean is

$$E\left(\bar{X}_{..}\right) = a + b\left(\frac{n-s}{2}\right) + bc_1 \tag{16}$$

where $E\left(\bar{e}_{..}\right) = 0$

2.2 Estimation of trend parameters and seasonal indices

The periodic and grand means are used to estimate parameters of trend line. The length of periodic interval is taken to be s. using the expression in (7) and (9), we obtain both mixed and multiplicative models as;

$$\bar{X}_{i.} = a - b(s - c_1) + (bs)i \tag{17}$$

$$\equiv \alpha + \beta_i \tag{18}$$

Hence, $\hat{a} = \alpha + \hat{b}(s - c_1)$ (19)

$$\hat{b} = \frac{\beta}{s} \tag{20}$$

For mixed model, when b=0, that is when there is no trend,

$$\bar{X}_{i.} = a + \bar{e}_i \tag{21}$$

For multiplicative model, when b=0, that is when there is no trend,

UNDER PEER REVIEW

$$\bar{X}_{i.} = a \tag{22}$$

Estimation of $S_j, \quad j = 1,2,...,s$

The seasonal and grand means are used to estimate the seasonal indices. The length of periodic interval is also taken to be s. using the expression in (11) and (13), we obtain both mixed and multiplicative models

$$\bar{X}_j = \left[a + b\left(\frac{n-s}{2}\right) + bj\right] S_j \tag{23}$$

$$\equiv \lfloor \alpha + \beta_j \rfloor S_j \tag{24}$$

Where. $\alpha = a + b\left(\frac{n-s}{2}\right)$ (25)

$$\beta = b \tag{26}$$

$$\therefore \quad S_j = \frac{\bar{X}_j}{a + b\left(\frac{n-s}{2}\right) + b_j} \tag{27}$$

For mixed model, where there is no trend (b=0), we obtain from (11)

$$\hat{S}_j = \frac{\bar{X}_j}{a + \bar{e}_j} \tag{28}$$

For multiplicative model, when b=0, that is when there is no trend, we obtain from (13)

$$\hat{S}_j = \frac{\bar{X}_{.j}}{a\bar{e}_{.j}}$$

UNDER PEER REVIEW

Table 1: Expected values of means for multiplicative and mixed models

Linear trend-cycle component: $M_t = a + b_t, \quad t = 1,2,...,n = ms$

Measures	Multiplicative model	Mixed model
$\bar{X}_{i.}$	$[a - bs + bsi] + \frac{b}{s}\sum_{j=1}^{s} jS_j$	$a - bs + bsi + \frac{b}{s}\sum_{j=1}^{s} jS_j$
$\bar{X}_{.j}$	$\left[a + b\left(\frac{n-s}{2}\right) + bj\right]S_j$	$\left[a + b\left(\frac{n-s}{2}\right) + bj\right]S_j$
$\bar{X}_{..}$	$a + b\left(\frac{n-s}{2}\right) + bc_1$	$a + b\left(\frac{n-s}{2}\right) + bc_1$

Where, $c_1 = \frac{1}{s}\sum_{j=1}^{s} jS_j$

Table 2: Estimates of trend and seasonal indices

Parameter	Multiplicative model	Mixed model
a	$a + \hat{b}(s - c_1)$	$a + \hat{b}(s - c_1)$
b	$\frac{\beta}{s}$	$\frac{\beta}{s}$
S_j	$\frac{\bar{X}_{.j}}{a + b\left(\frac{n-s}{2}\right) + b_j}$	$\frac{\bar{X}_{.j}}{a + b\left(\frac{n-s}{2}\right) + b_j}$

UNDER PEER REVIEW

Table 3: Estimates of means and variances for mixed and multiplicative models

	Linear trend-cycle component: $M_t = a + bt$, $t = 1, 2, ..., n = ms$	
Measures	Multiplicative model	Mixed model
$\bar{X}_{i.}$	$\left[a - bs + \frac{b}{s}\sum_{j=1}^{s} jS_j + bsi\right] * \bar{e}_{i.}$	$[a - bs + bsi] + \frac{b}{s}\sum_{j=1}^{s} jS_j + \bar{e}_{i.}$
$\bar{X}_{.j}$	$\left[a\bar{e}_{.j} + \frac{bs}{m}\sum_{i=1}^{m} i\, e_{ij} - bs\bar{e}_{,j} + bj\bar{e}_{.j}\right] * S_j$	$\left[a + b\left(\frac{n-s}{2}\right) + bj\right] * S_j + \bar{e}_{.j}$
$\bar{X}_{..}$	$a + b\left(\frac{n-s}{2}\right) + bC_1$	$a + b\left(\frac{n-s}{2}\right) + bC_1 + \bar{e}_{..}$
$\hat{\sigma}^2_{i.}$	$\left\{[(a + bs(i-1)) + bC_1]^2 + var\left[\begin{matrix}[a + bs(i-1)]S_j \\ + bjS_j\end{matrix}\right]\right\} \sigma_2^2$	$\{[(a + bs(i-1)) + bC_1]^2 + var[[a + bs(i-1)]S_j + bjS_j]\} + \sigma_1^2$
$\hat{\sigma}^2_{.j.}$	$\left\{\frac{b^2(n^2 - s^2)}{12} + \left[a + b\left(\frac{n-s}{2}\right) + bj\right]^2\right\} S_j^2\ \sigma_2^2$	$\frac{b^2 n(n+s)}{12} S_j^2 + \sigma_1^2$
$\hat{\sigma}^2_{x}$	$\left\{\begin{matrix}\frac{b^2(n^2 - s^2)}{12} + \left[a + b\left(\frac{n-s}{2}\right) + C_1\right]^2 \\ + \left[a^2 + 2ab\left(\frac{n-s}{2}\right) + \frac{b^2(n-s)(2n-s)}{6}\right] Var(S_j) \\ + b^2\, Var(jS_j) + 2b\left[a + b\left(\frac{n-s}{2}\right)\right] Cov(S_j, jS_j)\end{matrix}\right\} \sigma_2^2$	$\frac{n}{n-1}\left\{\begin{matrix}\frac{b^2(n^2 - s^2)}{12} + \left[a^2 + 2ab\left(\frac{n-s}{2}\right) + \frac{b^2(n-s)(2n-s)}{6}\right] Var(S_j) \\ + 2b\left[a + b\left(\frac{n-s}{2}\right)\right] Cov(S_j, jS_j) + b^2\, Var(jS_j)\end{matrix}\right\} + \sigma_1^2$

UNDER PEER REVIEW

Table 4: Estimates of trend and seasonal indices (when there is no trend (b=0))

Parameter	Multiplicative model	Mixed model
$\bar{X}_{i.}$	a	$a+\bar{e}_{\cdot i}$
$\bar{X}_{.j}$	a	$a+\bar{e}_{\cdot i}$
$\bar{X}_{\cdot\cdot}$	a	$a+\bar{e}_{\cdot\cdot}$
S_j	$\frac{\bar{X}_{\cdot j}}{a\bar{e}_{\cdot j}}$	$\frac{\bar{X}_{\cdot j}}{a+\bar{e}_{\cdot j}}$

3. Real Life Data

The purpose of this section is to discuss real life example, based on monthly time series data on number of baptism collected from Assumpta Cathedral Owerri, Imo State, Nigeria for a period of 2009 to 2018 given in Appendix A. The time series plots of actual and transformed data sets are given in figure 1 and 2. The expression of linear trend and seasonal indices for both mixed and multiplicative models given as

$$\bar{X}_{.j} = 2.584 + 0.0201j \quad (29)$$

Using (25),(26) and (27)

$$\hat{b} = 0.0201, \qquad \hat{a} = 2.584 - 0.0201\left(\frac{120-12}{2}\right)$$

Table 5: Estimates of trend parameters

Parameter	Mixed model values	Multiplicative model values
a	1.4986	1.4986
b	0.0201	0.0201

UNDER PEER REVIEW

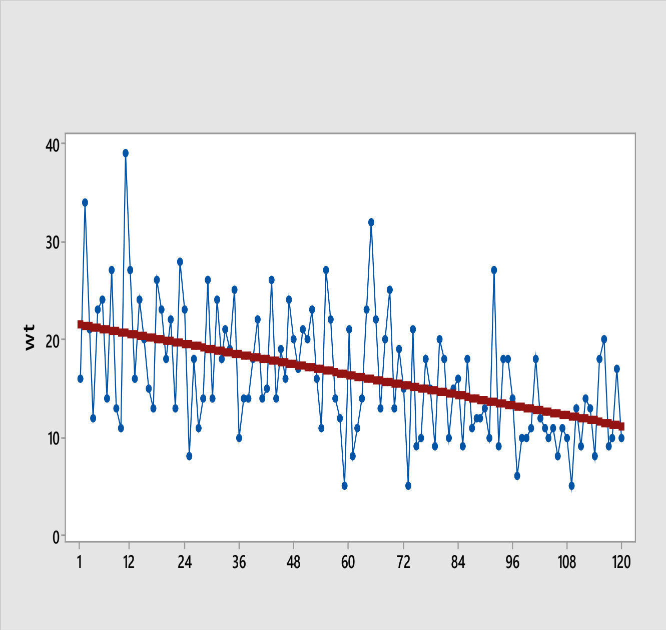


Figure 1:Time plot of the actual series

$$\hat{a} = 1.4986$$

$$\hat{S}_j = \frac{\bar{X}._j}{2.584 + 0.0201_j}$$

Table 6: Estimates of Seasonal indices

j	$\bar{X}_j$	$\hat{S}_j$
1	2.2380	0.8594
2	2.8480	1.0853
3	2.6060	0.9855
4	2.7018	1.0140
5	2.8670	1.0680
6	2.7110	1.0024
7	2.7860	1.0225
8	2.9392	1.0708
9	2.7190	0.9834
10	2.5305	0.9086
11	2.8780	1.0263
12	2.7500	0.9734
$\sum_{j=1}^{s} \hat{S}_j$		12.0000

UNDER PEER REVIEW

Table 7: Estimates of parameters of trend and seasonal indices

Parameter	Multiplicative model values	Mixed model values
$\hat{a}$	1.4986	1.4986
$\hat{b}$	0.0201	0.0201
$\hat{S}_1$	0.8594	0.8594
$\hat{S}_2$	1.0853	1.0853
$\hat{S}_3$	0.9855	0.9855
$\hat{S}_4$	1.0140	1.0140
$\hat{S}_5$	1.0680	1.0680
$\hat{S}_6$	1.0024	1.0024
$\hat{S}_7$	1.0225	1.0225
$\hat{S}_8$	1.0708	1.0708
$\hat{S}_9$	0.9834	0.9834
$\hat{S}_{10}$	0.9086	0.9086
$\hat{S}_{11}$	1.0263	1.0263
$\hat{S}_{12}$	0.9734	0.9734
$\sum_{j=1}^{s} \hat{S}_j$	12.0000	12.0000

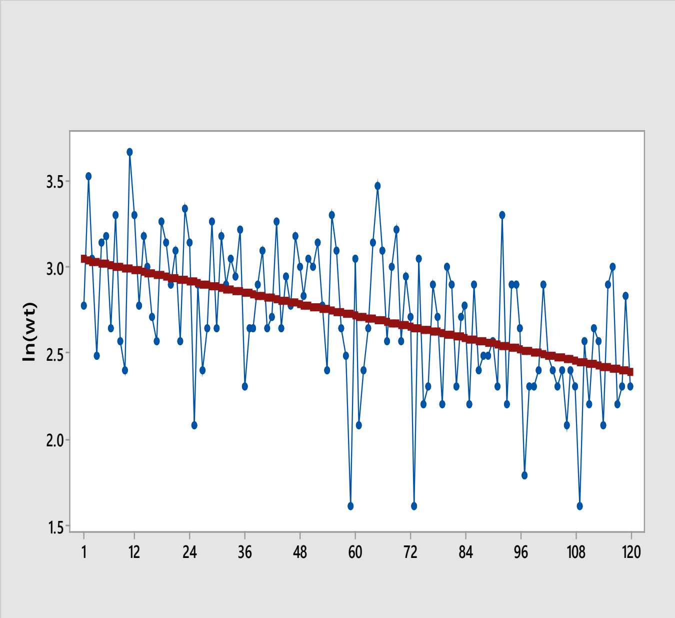
UNDER PEER REVIEW

Note: mixed satisfies

$\left(\sum_{j=1}^{s} S_j = s \right)$ as in (6)

Also, multiplicative model satisfies

$\left(\sum_{j=1}^{s} S_j = s \right)$ as in (6)



Time

Figure 2:Time plot of the transformed series

4.0 Concluding Remarks

Results of the study show that 1) the means and variances of the Buys-Ballot table for mixed and multiplicative models are not the same. 2) the expected values of the means are the same both for mixed and multiplicative models 3) the computed values of estimated trend parameters and seasonal indices are the same for the two models, but different when there is no trend. The column variances ($\hat{\sigma}_j^2$) of the Buys-Ballot table depends on the season j only through the square of the seasonal effect S_j^2 for mixed model. A quadratic function of the column j and square of the seasonal effect S_j^2 for multiplicative model.

REFERENCES

[1] Chatfield, C. (2004). *The analysis of time Series: An introduction*. Chapman and Hall,/CRC Press, Boca Raton.

[2] Oladugba, A.V., Ukaegbu, E.C., Udom, A. U., Madukaife, M. S., Ugah, T. E. &Sanni, S.S., (2014). *Principles of Applied Statistic,* University of Nigeria Press Limited.

[3] Nwogu, E.C, Iwueze, I.S. Dozie, K.C.N. & Mbachu, H.I (2019). Choice between mixed and multiplicative models in time series decomposition. International Journal of Statistics and Applications 9(5), 153-159

[4] Dozie, K.C.N, Nwogu, E.C, Nwanya, J.C (2020). Buys-Ballot technique for analysis of time Series model. International Journal of Scientific Research and Innovative Technology 7(1), 63-78

UNDER PEER REVIEW

[5] Wei, W. W. S (1989). *Time series analysis: Univariate and multivariate methods*, Addison-Wesley publishing Company Inc, Redwood City.

[6] Iwueze, I. S. & Nwogu E.C. (2004). *Buys-Ballot estimates for time series decomposition,* Global Journal of Mathematics, 3(2), 83-98.

[7] Iwueze, I. S. & Nwogu, E.C. (2005). *Buys-Ballot estimates for exponential and s-shaped curves, for time series,* Journal of the Nigerian Association of Mathematical Physics, 9, 357-366.

[8] Iwueze, I. S. & Nwogu, E.C. (2014).*Framework for choice of models and detection of seasonal effect in time series.*Far East Journal of Theoretical Statistics 48(1), 45– 66

[9] Iwueze, I. S. and Ohakwe, J. (2004). *Buys-Ballot estimates when stochastic trend is quadratic*. Journal of the Nigerian Association of Mathematical Physics, 8, 311-318.

[10] Dozie, K. C. N (2020). Buys-Ballot estimates for mixed model in descriptive time series. International Journal of Theoretical and Mathematical Physics 10(1), 22-27

[11] Dozie, K.C.N, Ijomah M.A (2020). A comparative study on additive and mixed models in descriptive time series. American Journal of Mathematical and Computer Modelling 5(1), 12-17

UNDER PEER REVIEW

Appendix A: Buys-Ballot table for the actual data on number of Registered Baptism in Assumpta Cathedral Parish Owerri (2009-2018)

Year	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sept.	Oct.	Nov.	Dec.	$\bar{X}_{i.}$	$\sigma^2_{i.}$
2009	16	34	21	12	23	24	14	27	13	11	39	27	21.75	80.93
2010	16	24	20	15	11	26	23	18	22	13	28	23	20.08	25.54
2011	8	18	11	14	26	14	24	18	21	19	25	10	17.33	36.24
2012	14	14	18	22	14	15	26	14	19	16	24	20	18.0	19.8
2013	17	21	20	23	16	11	27	22	14	12	5	21	17.42	37.72
2014	8	11	14	23	32	22	13	20	25	13	19	15	17.92	46.81
2015	5	21	9	10	18	15	9	20	18	10	15	16	13.83	25.95
2016	9	18	11	12	12	13	10	27	9	18	18	14	14.25	27.30
2017	6	10	10	11	18	12	11	10	11	8	11	10	10.67	7.88
2018	5	13	9	14	13	8	18	20	9	10	17	10	12.17	20.15
$\bar{X}_{.j}$	10.4	18.4	14.3	15.6	18.5	16.0	17.5	19.6	16.1	13.0	20.1	16.6	16.34	
$\sigma^2_{.j}$	23.38	51.38	24.46	26.04	50.78	35.56	47.39	27.16	32.22	13.11	90.1	35.6		41.37

Appendix B: Buys-Ballot table for the transformed data on number of Registered Baptism in Assumpta Cathedral Parish Owerri (2009-2018)

Year	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sept.	Oct.	Nov.	Dec.	$\bar{X}_{i.}$	$\sigma^2_{i.}$
2009	2.77	3.53	3.04	2.48	3.14	3.18	2.64	3.30	2.56	2.40	3.66	3.30	3.00	0.18
2010	2.77	3.18	2.30	2.71	2.56	3.26	3.14	2.89	3.09	2.56	3.33	3.14	2.91	0.11
201	2.0	2.8	2.40	2.6	3.2	2.6	3.1	2.8	3.04	2.9	3.2	2.3	2.7	0.1

1	8	9		4	6	4	8	9		4	2	0	9	5
2012	2.64	2.64	2.89	3.09	2.64	2.71	3.26	2.64	2.94	2.77	3.18	2.30	2.81	0.08
2013	2.83	3.04	2.30	3.14	2.77	2.40	3.30	3.09	2.64	2.48	1.61	3.04	2.72	0.22
2014	2.08	2.40	2.64	3.14	3.47	3.09	2.56	2.30	3.22	2.56	2.94	2.71	2.76	0.17
2015	1.61	3.04	2.20	2.30	2.89	2.71	2.20	2.30	2.08	2.30	2.71	2.77	2.43	0.16
2016	2.20	2.89	2.40	2.48	2.48	2.56	2.30	3.30	2.20	2.89	2.89	2.64	2.60	0.11
2017	1.79	2.30	2.30	2.40	2.89	2.48	2.40	2.30	2.40	2.08	2.40	2.30	2.34	0.06
2018	1.61	2.56	2.20	2.64	2.56	2.08	2.89	2.30	2.20	2.30	2.83	2.30	2.37	0.12
$\overline{X}_{.j}$	2.24	2.85	2.47	2.70	2.87	2.71	2.79	2.73	2.64	2.53	2.88	2.68	2.67	
$\sigma_{.j}^{2}$	0.24	0.14	0.09	0.10	0.11	0.14	0.18	0.18	0.17	0.08	0.32	0.15		0.13

UNDER PEER REVIEW