On Summing Formulas For Horadam Numbers

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Abstract. In this paper, closed forms of the summation formulas for generalized Fibonacci numbers are presented. As special cases, we give summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers.

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1. Introduction

Horadam [3] defined a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence $\{W_n(W_0, W_1; r, s)\}$, or simply $\{W_n\}$, as follows:

$$W_n = rW_{n-1} + sW_{n-2}; \quad W_0 = c, \ W_1 = d, \quad (n \geq 2) \tag{1.1}$$

where W_0, W_1 are arbitrary complex numbers and r, s are real numbers, see also Horadam [2], [4] and [5]. Now these generalized Fibonacci numbers $\{W_n(a, b; r, s)\}$ are also called Horadam numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

For some specific values of c, d, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences.

Name of sequence	Notation: $W_n(c,d;r,s)$	OEIS: [11]
Fibonacci	$F_n = W_n(0,1;1,1)$	A000045
Lucas	$L_n = W_n(2,1;1,1)$	A000032
Pell	$P_n = W_n(0,1;2,1)$	A000129
Pell-Lucas	$Q_n = W_n(2,2;2,1)$	A002203
Jacobsthal	$J_n = W_n(0,1;1,2)$	A001045
Jacobsthal-Lucas	$j_n = W_n(2,1;1,2)$	A014551

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

THEOREM 1.1. *For $n \geq 0$ we have the following formulas:*

(a): *(Sum of the generalized Fibonacci numbers) If $r+s-1 \neq 0$, then*

$$\sum_{i=0}^{n} W_i = \frac{W_{n+2} + (1-r)W_{n+1} - W_1 + (r-1)W_0}{r+s-1}.$$

(b): *If $(r-s+1)(r+s-1) \neq 0$ then*

$$\sum_{i=0}^{n} W_{2i} = \frac{(1-s)W_{2n+2} + rsW_{2n+1} + (s-1)W_2 - rsW_1 + (r^2-s^2+2s-1)W_0}{(r-s+1)(r+s-1)}.$$

(c): *If $(r-s+1)(r+s-1) \neq 0$ then*

$$\sum_{i=0}^{n} W_{2i+1} = \frac{rW_{2n+2} + (s-s^2)W_{2n+1} - rW_2 + (r^2+s-1)W_1}{(r-s+1)(r+s-1)}.$$

Proof. This is given in [12].

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

THEOREM 1.2. *For $n \geq 1$ we have the following formulas:*

(a): *(Sum of the generalized Fibonacci numbers with negative indices) If $r+s-1 \neq 0$, then*

$$\sum_{i=1}^{n} W_{-i} = \frac{-(r+s)W_{-n-1} - sW_{-n-2} + W_1 + (1-r)W_0}{r+s-1}.$$

(b): *If $(r-s+1)(r+s-1) \neq 0$ then*

$$\sum_{i=1}^{n} W_{-2i} = \frac{(s-1)W_{-2n} - rsW_{-2n-1} + rW_1 + (1-s-r^2)W_0}{(r-s+1)(r+s-1)}$$

(c): *If $(r-s+1)(r+s-1) \neq 0$ the*

$$\sum_{i=1}^{n} W_{-2i+1} = \frac{-rW_{-2n} + (s^2-s)W_{-2n-1} + (1-s)W_1 + rsW_0}{(r-s+1)(r+s-1)}.$$

Proof. This is given in [12].

In this work, we investigate some summation formulas of generalized Fibonaci numbers. We present some works on summing formulas of the numbers in the following Table 2.

Table 2. A few special study of sum formulas.

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[6], [8, 9]
Generalized Fibonacci	[7,12]
Generalized Tribonacci	[1,10,13,14]
Generalized Tetranacci	[15,16, 20]
Generalized Pentanacci	[17,18]
Generalized Hexanacci	[19]

2. Summing Formulas of Generalized Fibonacci Numbers with Positive Subscripts

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

THEOREM 2.1. *For $n \geq 0$ we have the following formulas:*

(a): *If $r+s-1 \neq 0$, then*

$$\sum_{i=0}^{n} iW_i = \frac{\Delta_1}{(r+s-1)^2}$$

where

$$\begin{aligned} \Delta_1 &= ((r-2)+(s+r-1)n)W_{n+2} + ((-r^2+2r-s-1)-(r-1)(r+s-1)n)W_{n+1} \\ &\quad +(1+s)W_1 + s(2-r)W_0. \end{aligned}$$

(b): *If $(r-s+1)(r+s-1) \neq 0$ then*

$$\sum_{i=0}^{n} iW_{2i} = \frac{\Delta_2}{(r-s+1)^2 (r+s-1)^2}$$

where

$$\begin{aligned} \Delta_2 &= ((1-s)(r-s+1)(r+s-1)n - (r^2s+s^2-2s+1))W_{2n+2} \\ &\quad +rs((r-s+1)(r+s-1)n + r^2 + 2s - 2)W_{2n+1} \\ &\quad +r(1-s^2)W_1 + s(r^2s+s^2-2s+1)W_0. \end{aligned}$$

(c): *If $(r-s+1)(r+s-1) \neq 0$ then*

$$\sum_{i=0}^{n} iW_{2i+1} = \frac{\Delta_3}{(r-s+1)^2 (r+s-1)^2}$$

where

$$\begin{aligned}\Delta_3 &= r((r-s+1)(r+s-1)n+s^2-1)W_{2n+2} \\ &+(s(1-s)(r+s-1)(r-s+1)n-s(r^2s+s^2-2s+1))W_{2n+1} \\ &+(s^3+r^2-2s^2+s)W_1+rs(1-s^2)W_0.\end{aligned}$$

Proof.

(a): Using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}$$

i.e.

$$sW_{n-2} = W_n - rW_{n-1}$$

we obtain

$$\begin{aligned} snW_n &= nW_{n+2}-rnW_{n+1} \\ s(n-1)W_{n-1} &= (n-1)W_{n+1}-r(n-1)W_n \\ s(n-2)W_{n-2} &= (n-2)W_n-r(n-2)W_{n-1} \\ s(n-3)W_{n-3} &= (n-3)W_{n-1}-r(n-3)W_{n-2} \\ &\vdots \\ s5W_5 &= 5W_7-r5W_6 \\ s4W_4 &= 4W_6-r5W_5 \\ s3W_3 &= 3W_5-r3W_4 \\ s2W_2 &= 2W_4-r2W_3 \\ sW_1 &= W_3-rW_2 \end{aligned}$$

If we add the equations by side by, we get

$$s\sum_{i=0}^{n} iW_i = \sum_{i=3}^{n+2}(i-2)W_i - r\sum_{i=2}^{n+1}(i-1)W_i \tag{2.1}$$

Note that

$$\begin{aligned}\sum_{i=3}^{n+2}(i-2)W_i &= W_1+2W_0+(n-1)W_{n+1}+nW_{n+2}+\sum_{i=0}^{n} iW_i-2\sum_{i=0}^{n} W_i \\ \sum_{i=2}^{n+1}(i-1)W_i &= W_0+nW_{n+1}+\sum_{i=0}^{n} iW_i-\sum_{i=0}^{n} W_i.\end{aligned}$$

If we put them in (2.1) then it follows that

$$(r+s-1)\sum_{i=0}^{n} iW_i = W_1+2W_0-rW_0+(n-rn-1)W_{n+1}+nW_{n+2}+(r-2)\sum_{i=0}^{n} W_i.$$

Then, if we use Theorem 1.1 (a), the required results of (a) follows.

(b) and (c): Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2}$$

we obtain

$$\begin{aligned}
rnW_{2n+1} &= nW_{2n+2} - snW_{2n} \\
r(n-1)W_{2n-1} &= (n-1)W_{2n} - s(n-1)W_{2n-2} \\
&\vdots \\
r4W_9 &= 4W_{10} - s4W_8 \\
r3W_7 &= 3W_8 - s3W_6 \\
r2W_5 &= 2W_6 - s2W_4 \\
rW_3 &= W_4 - sW_2 \\
r \times 0 \times W_1 &= 0 \times W_2 - s \times 0 \times W_0
\end{aligned}$$

Now, if we add the above equations by side by, we get

$$r\sum_{i=0}^{n} iW_{2i+1} = \sum_{i=1}^{n+1}(i-1)W_{2i} - s\sum_{i=0}^{n} iW_{2i} \tag{2.2}$$

Note that

$$\sum_{i=1}^{n+1}(i-1)W_{2i} = W_0 + nW_{2n+2} + \sum_{i=0}^{n}(i-1)W_{2i} = W_0 + nW_{2n+2} + \sum_{i=0}^{n} iW_{2i} - \sum_{i=0}^{n} W_{2i}.$$

If we put this in (2.2) we obtain

$$r\sum_{i=0}^{n} iW_{2i+1} = W_0 + nW_{2n+2} + (1-s)\sum_{i=0}^{n} iW_{2i} - \sum_{i=0}^{n} W_{2i} \tag{2.3}$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} \Rightarrow rW_n = W_{n+1} - sW_{n-1} \Rightarrow rW_{2n} = W_{2n+1} - sW_{2n-1}$$

we write the following obvious equations;

$$\begin{aligned}
r(n+1)W_{2n+2} &= (n+1)W_{2n+3} - s(n+1)W_{2n+1} \\
rnW_{2n} &= nW_{2n+1} - snW_{2n-1} \\
r(n-1)W_{2n-2} &= (n-1)W_{2n-1} - s(n-1)W_{2n-3} \\
&\vdots \\
r4W_8 &= 4W_9 - s4W_7 \\
r3W_6 &= 3W_7 - s3W_5 \\
r2W_4 &= 2W_5 - s2W_3 \\
rW_2 &= W_3 - sW_1 \\
r \times 0 \times W_0 &= 0 \times W_1 - s \times 0 \times W_{-1}
\end{aligned}$$

Now, if we add the above equations by side by, we obtain

$$r\sum_{i=0}^{n} iW_{2i} = \sum_{i=0}^{n} iW_{2i+1} - s\sum_{i=-1}^{n-1}(i+1)W_{2i+1}. \tag{2.4}$$

Note that

$$\sum_{i=-1}^{n-1}(i+1)W_{2i+1} = -(n+1)W_{2n+1} + \sum_{i=0}^{n} iW_{2i+1} + \sum_{i=0}^{n} W_{2i+1}$$

If we put this in (2.4) we obtain

$$r\sum_{i=0}^{n} iW_{2i} = (1-s)\sum_{i=0}^{n} iW_{2i+1} + s(n+1)W_{2n+1} - s\sum_{i=0}^{n} W_{2i+1}. \tag{2.5}$$

Then, using Theorem 1.1 (b) and (c) and

$$W_2 = (rW_1 + sW_0)$$

and solving the system (2.3)-(2.5), the required result of (b) and (c) follow.

Taking $r = s = 1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

Proposition 2.2. *If $r = s = 1$ then for $n \geq 0$ we have the following formulas:*

- **(a):** $\sum_{i=0}^{n} iW_i = (n-1)W_{n+2} - W_{n+1} + 2W_1 + W_0$.
- **(b):** $\sum_{i=0}^{n} iW_{2i} = -W_{2n+2} + (n+1)W_{2n+1} + W_0$.
- **(c):** $\sum_{i=0}^{n} iW_{2i+1} = nW_{2n+2} - W_{2n+1} + W_1$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 2.3. *For $n \geq 0$, Fibonacci numbers have the following properties:*

- **(a):** $\sum_{i=0}^{n} iF_i = (n-1)F_{n+2} - F_{n+1} + 2$.

(b): $\sum_{i=0}^{n} iF_{2i} = -F_{2n+2} + (n+1) F_{2n+1}$.

(c): $\sum_{i=0}^{n} iF_{2i+1} = nF_{2n+2} - F_{2n+1} + 1$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.4. *For $n \geq 0$, Lucas numbers have the following properties:*

(a): $\sum_{i=0}^{n} iL_i = (n-1) L_{n+2} - L_{n+1} + 4$.

(b): $\sum_{i=0}^{n} iL_{2i} = -L_{2n+2} + (n+1) L_{2n+1} + 2$.

(c): $\sum_{i=0}^{n} iL_{2i+1} = nL_{2n+2} - L_{2n+1} + 1$.

Taking $r = 2, s = 1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

Proposition 2.5. *If $r = 2, s = 1$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{i=0}^{n} iW_i = \frac{1}{2}(nW_{n+2} - (1+n)W_{n+1} + W_1)$.

(b): $\sum_{i=0}^{n} iW_{2i} = \frac{1}{4}(-W_{2n+2} + 2(n+1)W_{2n+1} + W_0)$.

(c): $\sum_{i=0}^{n} iW_{2i+1} = \frac{1}{4}(2nW_{2n+2} - W_{2n+1} + W_1)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 2.6. *For $n \geq 0$, Pell numbers have the following properties:*

(a): $\sum_{i=0}^{n} iP_i = \frac{1}{2}(nP_{n+2} - (1+n)P_{n+1} + 1)$.

(b): $\sum_{i=0}^{n} iP_{2i} = \frac{1}{4}(-P_{2n+2} + 2(n+1)P_{2n+1})$.

(c): $\sum_{i=0}^{n} iP_{2i+1} = \frac{1}{4}(2nP_{2n+2} - P_{2n+1} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 2.7. *For $n \geq 0$, Pell-Lucas numbers have the following properties:*

(a): $\sum_{i=0}^{n} iQ_i = \frac{1}{2}(nQ_{n+2} - (1+n)Q_{n+1} + 2)$.

(b): $\sum_{i=0}^{n} iQ_{2i} = \frac{1}{4}(-Q_{2n+2} + 2(n+1)Q_{2n+1} + 2)$.

(c): $\sum_{i=0}^{n} iQ_{2i+1} = \frac{1}{4}(2nQ_{2n+2} - Q_{2n+1} + 2)$.

If $r = 1, s = 2$ then $(r-s+1)(r+s-1) = 0$ so we can't use Theorem 2.1 (b) and (c), directly. However, we can find $\sum_{i=0}^{n} iW_{2i}$ and $\sum_{i=0}^{n} iW_{2i+1}$ using mathematical induction which is given in the following theorem.

Theorem 2.8. *If $r = 1, s = 2$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{i=0}^{n} iW_i = \frac{1}{4}((2n-1)W_{n+2} - 2W_{n+1} + 3W_1 + 2W_0)$.

(b): $\sum_{i=0}^{n} iW_{2i} = \frac{1}{54}((4+21n)W_{2n+2} - 2(10+3n)W_{2n+1} + 8(2W_1 - W_0) - 9(W_1 - 2W_0)n^2)$.

We now prove (b) by induction on n. If $n=1$ we that the sum formula reduces to the relation

$$W_2=\frac{1}{54}((4+21\times 1)W_4-2(10+3\times 1)W_3+8(2W_1-W_0)-9(W_1-2W_0)). \tag{2.8}$$

Since

$$\begin{aligned} W_2 &= (2W_0+W_1), \\ W_3 &= (2W_0+3W_1), \\ W_4 &= (6W_0+5W_1), \end{aligned}$$

(2.8) is true. Assume that the relation in (b) is true for $n=m$, i.e.,

$$\sum_{i=1}^{m} iW_{2i}=\frac{1}{54}((4+21m)W_{2m+2}-2(10+3m)W_{2m+1}+8(2W_1-W_0)-9(W_1-2W_0)m^2)$$

Then we get

$$\begin{aligned} \sum_{i=1}^{m+1} iW_{2i} &= (m+1)W_{2m+2}+\sum_{i=1}^{m} iW_{2i} \\ &= (m+1)W_{2m+2}+\frac{1}{54}((4+21m)W_{2m+2}-2(10+3m)W_{2m+1} \\ &\quad +8(2W_1-W_0)-9(W_1-2W_0)m^2) \\ &= \frac{1}{54}((58+75m)W_{2m+2}-2(10+3m)W_{2m+1}+8(2W_1-W_0)-9(W_1-2W_0)m^2) \\ &= \frac{1}{54}((58+75m)W_{2m+2}-2(10+3m)W_{2m+1}+9\,(W_1-2W_0)\,(1+2m) \\ &\quad +8(2W_1-W_0)-9(W_1-2W_0)(m+1)^2) \\ &= \frac{1}{54}((4+21(m+1))W_{2m+4}-2(10+3(m+1))W_{2m+3}+8(2W_1-W_0)-9(W_1-2W_0)(m+1)^2) \\ &= \frac{1}{54}((4+21(m+1))W_{2(m+1)+2}-2(10+3(m+1))W_{2(m+1)+1} \\ &\quad +8(2W_1-W_0)-9(W_1-2W_0)(m+1)^2) \end{aligned}$$

where

$$(58+75m)W_{2m+2}-2(10+3m)W_{2m+1}+27\,(W_1-2W_0)=(4+21(m+1))W_{2m+4}-2(10+3(m+1))W_{2m+3}. \tag{2.9}$$

Note that (2.9) can be proved by using Binet formula of W_n. Hence, the relation in (a) holds also for $n=m+1$.

(c): We now prove (c) by induction on n. If $n=1$ we see that the sum formula reduces to the relation

$$W_3=\frac{1}{54}((8+15)W_4+2(-20+21)W_3+16(2W_1-W_0)+9(W_1-2W_0)). \tag{2.10}$$

Since

$$\begin{aligned} W_3 &= (2W_0+3W_1) \\ W_4 &= (6W_0+5W_1) \end{aligned}$$

(2.10) is true. Assume that the relation in (c) is true for $n=m$, i.e.,

$$\sum_{i=0}^{m} iW_{2i+1} = \frac{1}{54}((8+15m)W_{2m+2}+2(-20+21m)W_{2m+1}+16(2W_1-W_0)+9(W_1-2W_0)m^2)$$

Then we get

$$\begin{aligned} \sum_{i=0}^{m+1} iW_{2i+1} &= (m+1)W_{2m+3}+\sum_{i=0}^{m} iW_{2i+1} \\ &= (m+1)W_{2m+3}+\frac{1}{54}((8+15m)W_{2m+2}+2(-20+21m)W_{2m+1} \\ &\quad +16(2W_1-W_0)+9(W_1-2W_0)m^2) \\ &= \frac{1}{54}(54(m+1)W_{2m+3}+(8+15m)W_{2m+2}+2(-20+21m)W_{2m+1}-9\left(2m+1\right)\left(W_1-2W_0\right) \\ &\quad +16(2W_1-W_0)+9(W_1-2W_0)(m+1)^2) \\ &= \frac{1}{54}((8+15(m+1))W_{2m+4}+2(-20+21(m+1))W_{2m+3} \\ &\quad +16(2W_1-W_0)+9(W_1-2W_0)(m+1)^2) \\ &= \frac{1}{54}((8+15(m+1))W_{2(m+1)+2}+2(-20+21(m+1))W_{2(m+1)+1} \\ &\quad +16(2W_1-W_0)+9(W_1-2W_0)(m+1)^2) \end{aligned}$$

where

$$\begin{aligned} &54(m+1)W_{2m+3}+(8+15m)W_{2m+2}+2(-20+21m)W_{2m+1}-9\left(2m+1\right)\left(W_1-2W_0\right) \\ &= (8+15(m+1))W_{2m+4}+2(-20+21(m+1))W_{2m+3}. \end{aligned} \tag{2.11}$$

(2.11) can be proved by using Binet formula of W_n. Hence, the relation in (c) holds also for $n=m+1$.

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n=J_n$ with $J_0=0, J_1=1$).

Corollary 2.9. *For $n\geq 0$, Jacobsthal numbers have the following property:*

- **(a):** $\sum_{i=0}^{n} iJ_i = \frac{1}{4}((2n-1)J_{n+2}-2J_{n+1}+3)$.
- **(b):** $\sum_{i=0}^{n} iJ_{2i} = \frac{1}{54}((4+21n)J_{2n+2}-2(10+3n)J_{2n+1}+16-9n^2)$.
- **(c):** $\sum_{i=0}^{n} iJ_{2i+1} = \frac{1}{54}((8+15n)J_{2n+2}+2(-20+21n)J_{2n+1}+32+9n^2)$.

Taking $W_n=j_n$ with $j_0=2, j_1=1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.10. *For $n \geq 0$, Jacobsthal-Lucas numbers have the following property:*

(a): $\sum_{i=0}^{n} ij_i = \frac{1}{4}((2n-1)j_{n+2} - 2j_{n+1} + 7)$.

(b): $\sum_{i=0}^{n} ij_{2i} = \frac{1}{54}((4+21n)j_{2n+2} - 2(10+3n)j_{2n+1} + 27n^2)$.

(c): $\sum_{i=0}^{n} ij_{2i+1} = \frac{1}{54}((8+15n)j_{2n+2} + 2(-20+21n)j_{2n+1} - 27n^2)$.

3. Summing Formulas of Generalized Fibonacci Numbers with Negative Subscripts

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.1. *For $n \geq 1$ we have the following formulas:*

(a): *If $r+s-1 \neq 0$, then*

$$\sum_{i=1}^{n} iW_{-i} = \frac{\Delta_4}{(r+s-1)^2}$$

where

$$\begin{aligned}\Delta_4 &= -s\left((r+s-1)n + r + 2s\right)W_{-n-2} - ((r+s-1)(r+s)n \\ &\quad +(r+s)^2 + s)W_{-n-1} + (1+s)W_1 + s(2-r)W_0.\end{aligned}$$

(b): *If $(r-s+1)(r+s-1) \neq 0$ then*

$$\sum_{i=1}^{n} iW_{-2i} = \frac{\Delta_5}{(r-s+1)^2 (r+s-1)^2}$$

where

$$\begin{aligned}\Delta_5 &= ((s-1)(r-s+1)(r+s-1)n - (s^3 + r^2 - 2s^2 + s))W_{-2n} \\ &\quad +rs(-(r-s+1)(r+s-1)n + s^2 - 1)W_{-2n-1} \\ &\quad +r(1-s^2)W_1 + s(r^2 s + s^2 - 2s + 1)W_0.\end{aligned}$$

(c): *If $(r-s+1)(r+s-1) \neq 0$ then*

$$\sum_{i=1}^{n} iW_{-2i+1} = \frac{\Delta_6}{(r-s+1)^2 (r+s-1)^2}$$

where

$$\begin{aligned}\Delta_6 &= (-r(r-s+1)(r+s-1)n + (2s^2 - r^2 - 2s)r)W_{-2n} \\ &\quad +s((s-1)(r-s+1)(r+s-1)n + 2s^2 - r^2 - s^3 - s)W_{-2n-1} \\ &\quad +(s^3 + r^2 - 2s^2 + s)W_1 + rs(1-s^2)W_0.\end{aligned}$$

Proof.

(a): Using the recurrence relation

$$\begin{aligned} W_{-n+2} &= r\times W_{-n+1}+s\times W_{-n} \\ &\Rightarrow W_{-n}=-\frac{r}{s}\times W_{-n+1}+\frac{1}{s}W_{-n+2} \\ &\Rightarrow W_{-n}=-\frac{r}{s}\times W_{-(n-1)}+\frac{1}{s}W_{-(n-2)} \end{aligned}$$

i.e.

$$sW_{-n}=W_{-n+2}-rW_{-n+1}$$

or

$$W_{-n}=\frac{1}{s}W_{-n+2}-\frac{r}{s}W_{-n+1}$$

we obtain

$$\begin{aligned} s\times(n+2)\times W_{-n-2} &= (n+2)\times W_{-n}-r\times(n+2)\times W_{-n-1} \\ s\times(n+1)\times W_{-n-1} &= (n+1)\times W_{-n+1}-r\times(n+1)\times W_{-n} \\ s\times n\times W_{-n} &= n\times W_{-n+2}-r\times n\times W_{-n+1} \\ s\times(n-1)\times W_{-n+1} &= (n-1)\times W_{-n+3}-r\times(n-1)\times W_{-n+2} \\ s\times(n-2)\times W_{-n+2} &= (n-2)\times W_{-n+4}-r\times(n-2)\times W_{-n+3} \\ &\vdots \\ s\times 5\times W_{-5} &= 5\times W_{-3}-r\times 5\times W_{-4} \\ s\times 4\times W_{-4} &= 4\times W_{-2}-r\times 4\times W_{-3} \\ s\times 3\times W_{-3} &= 3\times W_{-1}-r\times 3\times W_{-2} \\ s\times 2\times W_{-2} &= 2\times W_{0}-r\times 2\times W_{-1} \\ s\times 1\times W_{-1} &= 1\times W_{1}-r\times 1\times W_{0}. \end{aligned}$$

If we add the equations by side by, we get

$$s((n+1)W_{-n-1}+(n+2)W_{-n-2}+\sum_{i=1}^{n}iW_{-i})=(W_1+2W_0+\sum_{i=1}^{n}(i+2)W_{-i})-r((n+2)W_{-n-1}+W_0+\sum_{i=1}^{n}(i+1)W_{-i})$$

Note that since

$$\begin{aligned} \sum_{i=1}^{n}(i+2)W_{-i} &= \sum_{i=1}^{n}iW_{-i}+2\sum_{i=1}^{n}W_{-i}, \\ \sum_{i=1}^{n}(i+1)W_{-i} &= \sum_{i=1}^{n}iW_{-i}+\sum_{i=1}^{n}W_{-i}, \end{aligned}$$

we have

$$\begin{aligned} & s((n+1)W_{-n-1}+(n+2)W_{-n-2}+\sum_{i=1}^{n} iW_{-i}) \\ = & (W_1+2W_0+(\sum_{i=1}^{n} iW_{-i}+2\sum_{i=1}^{n} W_{-i}))-r((n+2)W_{-n-1}+W_0+(\sum_{i=1}^{n} iW_{-i}+\sum_{i=1}^{n} W_{-i})). \end{aligned}$$

Then, using Theorem 1.1 (a), the required results of (a) follows.

(b) and (c): Using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}$$

i.e.

$$rW_{-n+1}=W_{-n+2}-sW_{-n}$$

we obtain

$$\begin{aligned} r\times(n+1)\times W_{-2n-1} &= (n+1)\times W_{-2n}-s\times(n+1)\times W_{-2n-2} \\ r\times n\times W_{-2n+1} &= n\times W_{-2n+2}-s\times n\times W_{-2n} \\ r\times(n-1)\times W_{-2n+3} &= (n-1)\times W_{-2n+4}-s\times(n-1)\times W_{-2n+2} \\ r\times(n-2)\times W_{-2n+5} &= (n-2)\times W_{-2n+6}-s\times(n-2)\times W_{-2n+4} \\ &\vdots \\ r\times 3\times W_{-5} &= 3\times W_{-4}-s\times 3\times W_{-6} \\ r\times 2\times W_{-3} &= 2\times W_{-2}-s\times 2\times W_{-4} \\ r\times 1\times W_{-1} &= 1\times W_{0}-s\times 1\times W_{-2}. \end{aligned}$$

If we add the equations by side by, we get

$$r\sum_{i=1}^{n} iW_{-2i+1}=(-(n+1)W_{-2n}+W_0+\sum_{i=1}^{n}(i+1)W_{-2i})-s\sum_{i=1}^{n} iW_{-2i}.$$

Since

$$\sum_{i=1}^{n}(i+1)W_{-2i}=\sum_{i=1}^{n} iW_{-2i}+\sum_{i=1}^{n} W_{-2i}$$

it follows that

$$r\sum_{i=1}^{n} iW_{-2i+1}=-(n+1)W_{-2n}+W_0+(1-s)\sum_{i=1}^{n} iW_{-2i}+\sum_{i=1}^{n} W_{-2i}. \tag{3.1}$$

Similarly, using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}$$

i.e.

$$\begin{aligned} rW_{-n+1} &= W_{-n+2}-sW_{-n} \Rightarrow rW_{-2n+1}=W_{-2n+2}-sW_{-2n} \\ &\Rightarrow rW_{-2n+1-1}=W_{-2n+2-1}-sW_{-2n-1} \\ &\Rightarrow rW_{-2n}=W_{-2n+1}-sW_{-2n-1} \end{aligned}$$

we obtain

$$\begin{aligned} r\times n\times W_{-2n} &= n\times W_{-2n+1}-s\times n\times W_{-2n-1} \\ r\times(n-1)\times W_{-2n+2} &= (n-1)\times W_{-2n+3}-s\times(n-1)\times W_{-2n+1} \\ r\times(n-2)\times W_{-2n+4} &= (n-2)\times W_{-2n+5}-s\times(n-2)\times W_{-2n+3} \\ r\times(n-3)\times W_{-2n+6} &= (n-3)\times W_{-2n+7}-s\times(n-3)\times W_{-2n+5} \\ &\vdots \\ r\times 4\times W_{-8} &= 4\times W_{-7}-s\times 4\times W_{-9} \\ r\times 3\times W_{-6} &= 3\times W_{-5}-s\times 3\times W_{-7} \\ r\times 2\times W_{-4} &= 2\times W_{-3}-s\times 2\times W_{-5} \\ r\times 1\times W_{-2} &= 1\times W_{-1}-s\times 1\times W_{-3} \\ r\times 0\times W_{0} &= 0\times W_{1}-s\times 0\times W_{-1} \\ r\times(-1)\times W_{2} &= (-1)\times W_{3}-t\times(-1)\times W_{1} \end{aligned}$$

If we add the equations by side by, we get

$$r\sum_{i=1}^{n} iW_{-2i}=(\sum_{i=1}^{n} iW_{-2i+1})-s(nW_{-2n-1}+\sum_{i=1}^{n}(i-1)W_{-2i+1}).$$

Since

$$\sum_{i=1}^{n}(i-1)W_{-2i+1}=\sum_{i=1}^{n} iW_{-2i+1}-\sum_{i=1}^{n} W_{-2i+1}$$

it follows that

$$r\sum_{i=1}^{n} iW_{-2i}=-snW_{-2n-1}+(1-s)\sum_{i=1}^{n} iW_{-2i+1}+s\sum_{i=1}^{n} W_{-2i+1}. \tag{3.2}$$

Then, using Theorem 1.2 (b) and (c) and solving system (3.1)-(3.2) the required result of (b) and (c) follow.

Taking $r=s=1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

PROPOSITION 3.2. *If $r=s=1$ then for $n\geq 1$ we have the following formulas:*

(a): $\sum_{i=1}^{n} iW_{-i}=-(2n+5)W_{-n-1}-(n+3)W_{-n-2}+2W_1+W_0.$

(b): $\sum_{i=1}^{n} iW_{-2i}=-W_{-2n}-nW_{-2n-1}+W_0.$

(c): $\sum_{i=1}^{n} iW_{-2i+1} = -(n+1)W_{-2n} - W_{-2n-1} + W_1.$

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

COROLLARY 3.3. *For $n \geq 1$, Fibonacci numbers have the following properties.*

(a): $\sum_{i=1}^{n} iF_{-i} = -(2n+5)F_{-n-1} - (n+3)F_{-n-2} + 2.$
(b): $\sum_{i=1}^{n} iF_{-2i} = -F_{-2n} - nF_{-2n-1}.$
(c): $\sum_{i=1}^{n} iF_{-2i+1} = -(n+1)F_{-2n} - F_{-2n-1} + 1.$

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

COROLLARY 3.4. *For $n \geq 1$, Lucas numbers have the following properties.*

(a): $\sum_{i=1}^{n} iL_{-i} = -(2n+5)L_{-n-1} - (n+3)L_{-n-2} + 4.$
(b): $\sum_{i=1}^{n} iL_{-2i} = -L_{-2n} - nL_{-2n-1} + 2.$
(c): $\sum_{i=1}^{n} iL_{-2i+1} = -(n+1)L_{-2n} - L_{-2n-1} + 1.$

Taking $r = 2, s = 1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

PROPOSITION 3.5. *If $r = 2, s = 1$ then for $n \geq 1$ we have the following formulas:*

(a): $\sum_{i=1}^{n} iW_{-i} = \frac{1}{2}(-(5+3n)W_{-n-1} - (2+n)W_{-n-2} + W_1).$
(b): $\sum_{i=1}^{n} iW_{-2i} = \frac{1}{4}(-W_{-2n} - 2nW_{-2n-1} + W_0).$
(c): $\sum_{i=1}^{n} iW_{-2i+1} = \frac{1}{4}(-2(1+n)W_{-2n} - W_{-2n-1} + W_1).$

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

COROLLARY 3.6. *For $n \geq 1$, Pell numbers have the following properties.*

(a): $\sum_{i=1}^{n} iP_{-i} = \frac{1}{2}(-(5+3n)P_{-n-1} - (2+n)P_{-n-2} + 1).$
(b): $\sum_{i=1}^{n} iP_{-2i} = \frac{1}{4}(-P_{-2n} - 2nP_{-2n-1}).$
(c): $\sum_{i=1}^{n} iP_{-2i+1} = \frac{1}{4}(-2(1+n)P_{-2n} - P_{-2n-1} + 1).$

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

COROLLARY 3.7. *For $n \geq 1$, Pell-Lucas numbers have the following properties.*

(a): $\sum_{i=1}^{n} iQ_{-i} = \frac{1}{2}(-(5+3n)Q_{-n-1} - (2+n)Q_{-n-2} + 2).$
(b): $\sum_{i=1}^{n} iQ_{-2i} = \frac{1}{4}(-Q_{-2n} - 2nQ_{-2n-1} + 2).$
(c): $\sum_{i=1}^{n} iQ_{-2i+1} = \frac{1}{4}(-2(1+n)Q_{-2n} - Q_{-2n-1} + 2).$

If $r=1,s=2$ then $(r-s+1)(r+s-1)=0$ so we can't use Theorem 3.1 (b) and (c), directly. However, we can find $\sum_{i=1}^{n} iW_{-2i}$ and $\sum_{i=1}^{n} iW_{-2i+1}$ using mathematical induction which is given in the following theorem.

THEOREM 3.8. *If $r=0,s=2,t=1$ then for $n\geq 1$ we have the following formulas:*

(a): $\sum_{i=1}^{n} iW_{-i}=\frac{1}{4}(-(6n+11)W_{-n-1}-2(2n+5)W_{-n-2}+3W_1+2W_0)$.

(b): $\sum_{i=1}^{n} iW_{-2i}=\frac{1}{54}(-(8+3n)W_{-2n}-2(16+15n)W_{-2n-1}+8(2W_1-W_0)-9(W_1-2W_0)n^2)$.

(c): $\sum_{i=1}^{n} iW_{-2i+1}=\frac{1}{54}(-(16+33n)W_{-2n}-2(32+3n)W_{-2n-1}+16(2W_1-W_0)+9(W_1-2W_0)n^2)$.

Proof. (b) and (c) can be proved by mathematical induction.

(a): Taking $r=1,s=2$ in Theorem 3.1 (a) we obtain (a).

(b): The proof will be by induction on n. If $n=1$ we see that the sum formula reduces to the relation

$$W_{-2}=\frac{1}{54}(-(8+3)W_{-2}-2(16+15)W_{-3}+8(2W_1-W_0)-9(W_1-2W_0)). \tag{3.3}$$

Since

$$\begin{aligned} W_{-2} &= (\frac{3}{4}W_0-\frac{1}{4}W_1) \\ W_{-3} &= (-\frac{5}{8}W_0+\frac{3}{8}W_1) \end{aligned}$$

(3.3) is true. Assume that the relation in (b) is true for $n=m$, i.e.,

$$\sum_{i=1}^{m} iW_{-2i}=\frac{1}{54}(-(8+3m)W_{-2m}-2(16+15m)W_{-2m-1}+8(2W_1-W_0)-9(W_1-2W_0)m^2).$$

Then we get

$$\begin{aligned} \sum_{i=1}^{m+1} iW_{-2i} &= (m+1)W_{-2(m+1)}+\sum_{i=1}^{m} iW_{-2i} \\ &= (m+1)W_{-2m-2}+\frac{1}{54}(-(8+3m)W_{-2m}-2(16+15m)W_{-2m-1} \\ &\quad +8(2W_1-W_0)-9(W_1-2W_0)m^2) \\ &= \frac{1}{54}(-(8+3m)W_{-2m}-2(16+15m)W_{-2m-1}+54(m+1)W_{-2m-2} \\ &\quad +9(W_1-2W_0)(2m+1)+8(2W_1-W_0)-9(W_1-2W_0)(m+1)^2) \\ &= \frac{1}{54}(-(8+3(m+1))W_{-2m-2}-2(16+15(m+1))W_{-2m-3} \\ &\quad +8(2W_1-W_0)-9(W_1-2W_0)(m+1)^2) \\ &= \frac{1}{54}(-(8+3(m+1))W_{-2(m+1)}-2(16+15(m+1))W_{-2(m+1)-1} \\ &\quad +8(2W_1-W_0)-9(W_1-2W_0)(m+1)^2) \end{aligned}$$

where

$$-(8+3m)W_{-2m}-2(16+15m)W_{-2m-1}+54(m+1)W_{-2m-2}+9\left(W_1-2W_0\right)(2m+1) \tag{3.4}$$
$$= -(8+3(m+1))W_{-2m-2}-2(16+15(m+1))W_{-2m-3}.$$

(3.4) can be proved by using Binet formula of W_n. Hence, the relation in (b) holds also for $n=m+1$.

(c): We now prove (c) by induction on n. If $n=1$ we see that the sum formula reduces to the relation

$$W_{-1}=\frac{1}{54}(-(16+33\times 1)W_{-2}-2(32+3\times 1)W_{-3}+16(2W_1-W_0)+9(W_1-2W_0)\times 1^2) \tag{3.5}$$

Since

$$\begin{aligned} W_{-1} &= (-\frac{1}{2}W_0+\frac{1}{2}W_1) \\ W_{-2} &= (\frac{3}{4}W_0-\frac{1}{4}W_1) \\ W_{-3} &= (-\frac{5}{8}W_0+\frac{3}{8}W_1) \end{aligned}$$

(3.5) is true. Assume that the relation in (c) is true for $n=m$ i,e.,

$$\sum_{i=1}^{m} iW_{-2i+1}=\frac{1}{54}(-(16+33m)W_{-2m}-2(32+3m)W_{-2m-1}+16(2W_1-W_0)+9(W_1-2W_0)m^2).$$

Then we get

$$\begin{aligned} \sum_{i=1}^{m+1} iW_{-2i+1} &= (m+1)W_{-2m-1}+\sum_{i=1}^{m} iW_{-2i+1} \\ &= (m+1)W_{-2m-1}+\frac{1}{54}(-(16+33m)W_{-2m}-2(32+3m)W_{-2m-1} \\ &\quad +16(2W_1-W_0)+9(W_1-2W_0)m^2) \\ &= \frac{1}{54}(-(16+33m)W_{-2m}+2\left(24m-5\right)W_{-2m-1}-9\left(2m+1\right)\left(W_1-2W_0\right) \\ &\quad +16(2W_1-W_0)+9(W_1-2W_0)(m+1)^2) \\ &= \frac{1}{54}(-(16+33(m+1))W_{-2m-2}-2(32+3(m+1))W_{-2m-3}+16(2W_1-W_0) \\ &\quad +9(W_1-2W_0)(m+1)^2) \\ &= \frac{1}{54}(-(16+33(m+1))W_{-2(m+1)}-2(32+3(m+1))W_{-2(m+1)-1}+16(2W_1-W_0) \\ &\quad +9(W_1-2W_0)(m+1)^2) \end{aligned}$$

where

$$-(16+33m)W_{-2m}+2\left(24m-5\right)W_{-2m-1}-9\left(2m+1\right)\left(W_1-2W_0\right) \tag{3.6}$$
$$= -(16+33(m+1))W_{-2m-2}-2(32+3(m+1))W_{-2m-3}.$$

(3.6) can be proved by using Binet formula of W_n. Hence, the relation in (c) holds also for $n=m+1$.

From the last theorem, we have the following corollary which gives sum formula of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 3.9. *For $n \geq 1$, Jacobsthal numbers have the following property:*

- **(a):** $\sum_{i=1}^{n} iJ_{-i} = \frac{1}{4}(-(6n+11)J_{-n-1} - 2(2n+5)J_{-n-2} + 3)$.
- **(b):** $\sum_{i=0}^{n} iJ_{-2i} = \frac{1}{54}(-(8+3n)J_{-2n} - 2(16+15n)J_{-2n-1} + 16 - 9n^2)$.
- **(c):** $\sum_{i=0}^{n} iJ_{-2i+1} = \frac{1}{54}(-(16+33n)J_{-2n} - 2(32+3n)J_{-2n-1} + 32 + 9n^2)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.10. *For $n \geq 1$, Jacobsthal-Lucas numbers have the following property:*

- **(a):** $\sum_{i=1}^{n} ij_{-i} = \frac{1}{4}(-(6n+11)j_{-n-1} - 2(2n+5)j_{-n-2} + 7)$.
- **(b):** $\sum_{i=0}^{n} ij_{-2i} = \frac{1}{54}(-(8+3n)j_{-2n} - 2(16+15n)j_{-2n-1} + 27n^2)$.
- **(c):** $\sum_{i=0}^{n} ij_{-2i+1} = \frac{1}{54}(-(16+33n)j_{-2n} - 2(32+3n)j_{-2n-1} - 27n^2)$.

References

[1] Frontczak, R., Sums of Tribonacci and Tribonacci-Lucas Numbers, International Journal of Mathematical Analysis, 12 (1), 19-24, 2018.

[2] Horadam, A.F., A Generalized Fibonacci Sequence, American Mathematical Monthly, Vol. 68, pp. 455-459, 1961.

[3] Horadam, A. F., Basic Properties of a Certain Generalized Sequence of Numbers, Fibonacci Quarterly 3.3, 161-176, 1965.

[4] Horadam, A. F., Special Properties of The Sequence $w_n(a, b; p, q)$, Fibonacci Quarterly, Vol. 5, No. 5, pp. 424-434, 1967.

[5] Horadam, A. F., Generating functions for powers of a certain generalized sequence of numbers. Duke Math. J 32, 437-446, 1965.

[6] Gökbaş, H., Köse, H., Some sum formulas for products of Pell and Pell-Lucas numbers, Int. J. Adv. Appl. Math. and Mech. 4(4), 1-4, 2017.

[7] Hansen., R.T., General Identities for Linear Fibonacci and Lucas Summations, Fibonacci Quarterly, 16(2), 121-28, 1978.

[8] Koshy, T., Fibonacci and Lucas Numbers with Applications, A Wiley-Interscience Publication, New York, 2001.

[9] Koshy, T., Pell and Pell-Lucas Numbers with Applications, Springer, New York, 2014.

[10] Parpar, T., k'ncı Mertebeden Rekürans Bağıntısının Özellikleri ve Bazı Uygulamaları, Selçuk Üniversitesi, Fen Bilimleri Enstitüsü, Yüksek Lisans Tezi, 2011.

[11] Sloane, N.J.A., The on-line encyclopedia of integer sequences. Available: http://oeis.org/

[12] Soykan, Y., On Summing Formulas For Generalized Fibonacci and Gaussian Generalized Fibonacci Numbers, Advances in Research, 20(2), 1-15, 2019.

[13] Soykan, Y., Summing Formulas For Generalized Tribonacci Numbers, arXiv:1910.03490v1 [math.GM], 2019.

[14] Soykan, Y., Summing Formulas For Generalized Tribonacci Numbers, Universal Journal of Mathematics and Applications, accepted.

[15] Soykan, Y., Matrix Sequences of Tribonacci and Tribonacci-Lucas Numbers, arXiv:1809.07809v1 [math.NT] 20 Sep 2018.

[16] Soykan, Summation Formulas For Generalized Tetranacci Numbers, Asian Journal of Advanced Research and Reports, 7(2), 1-12, 2019. doi.org/10.9734/ajarr/2019/v7i230170.

[17] Soykan, Y., Linear Summing Formulas of Generalized Pentanacci and Gaussian Generalized Pentanacci Numbers, Journal of Advanced in Mathematics and Computer Science, 33(3): 1-14, 2019.

[18] Soykan, Y., Sum Formulas For Generalized Fifth-Order Linear Recurrence Sequences, Journal of Advances in Mathematics and Computer Science, 34(5), 1-14, 2019; Article no.JAMCS.53303, ISSN: 2456-9968, DOI: 10.9734/JAMCS/2019/v34i530224.

[19] Soykan, Y., On Summing Formulas of Generalized Hexanacci and Gaussian Generalized Hexanacci Numbers, Asian Research Journal of Mathematics, 14(4), 1-14, 2019; Article no.ARJOM.50727.

[20] Waddill, M. E., The Tetranacci Sequence and Generalizations, Fibonacci Quarterly, 9-20, 1992.