Original Research Article

# Properties of Generalized 6-primes Numbers

**Abstract.** In this paper, we introduce the generalized 6-primes sequence and we deal with, in detail, three special cases which we call them 6-primes, Lucas 6-primes and modified 6-primes sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

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**Keywords.** Hexanacci numbers, 6-primes numbers, Lucas 6-primes numbers, modified 6-primes numbers.

## 1. Introduction

In this paper, we investigate the generalized 6-primes sequences and we investigate, in detail, three special cases which we call them 6-primes, Lucas-6-primes and modified 6-primes sequences.

The sequence of Fibonacci numbers $\{F_n\}$ and the sequence of Lucas numbers $\{L_n\}$ are defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \qquad F_0 = 0, \; F_1 = 1,$$

and

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \qquad L_0 = 2, \; L_1 = 1$$

respectively. The generalizations of Fibonacci and Lucas sequences produce several nice and interesting sequences.

The generalized Hexanacci sequence $\{W_n(W_0, W_1, W_2, W_3, W_4, W_5; r_1, r_2, r_3, r_4, r_5, r_6)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined by the sixth-order recurrence relations

$$\begin{aligned} W_n &= r_1W_{n-1} + r_2W_{n-2} + r_3W_{n-3} + r_4W_{n-4} + r_5W_{n-5} + r_6W_{n-6}, \qquad (1.1) \\ W_0 &= c_0, W_1 = c_1, W_2 = c_2, \; W_3 = c_3, W_4 = c_4, W_5 = c_5, \; n \geq 6 \end{aligned}$$

where $W_0, W_1, W_2, W_3, W_4, W_5$ are arbitrary real or complex numbers and $r_1, r_2, r_3, r_4, r_5, r_6$ are real numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r_5}{r_6}W_{-n+1} - \frac{r_4}{r_6}W_{-n+2} - \frac{r_3}{r_6}W_{-n+3} - \frac{r_2}{r_6}W_{-n+4} - \frac{r_1}{r_6}W_{-n+5} + \frac{1}{r_6}W_{-n+6}$$

for $n = 1, 2, 3, ...$ when $r_6 \neq 0$. Therefore, recurrence (1.1) holds for all integer $n$. Hexanacci sequence has been studied by many authors, see for example [3], [4].

As $\{W_n\}$ is a sith order recurrence sequence (difference equation), it's characteristic equation is

$$x^6 - r_1x^5 - r_2x^4 - r_3x^3 - r_4x^2 - r_5x - r_6 = 0 \tag{1.2}$$

whose roots are $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$. Generalized Hexanacci numbers can be expressed, for all integers $n$, using Binet's formula.

THEOREM 1. *(Binet formula of generalized Hexanacci numbers)*

$$W_n = \sum_{k=1}^{6} \frac{b_k\alpha_k^n}{\prod\limits_{\substack{j=1 \\ k\neq j}}^{6}(\alpha_k - \alpha_j)} \tag{1.3}$$

*where*

$$\begin{aligned}
b_1 &= W_5 - (\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)W_4 + (\alpha_2\alpha_5+\alpha_2\alpha_3+\alpha_2\alpha_6+\alpha_5\alpha_3+\alpha_5\alpha_6+\alpha_2\alpha_4+\alpha_5\alpha_4+\alpha_3\alpha_6+\alpha_3\alpha_4+\alpha_6\alpha_4)W_3 \\
&\quad -(\alpha_2\alpha_5\alpha_3+\alpha_2\alpha_5\alpha_6+\alpha_2\alpha_5\alpha_4+\alpha_2\alpha_3\alpha_6+\alpha_5\alpha_3\alpha_6+\alpha_2\alpha_3\alpha_4+\alpha_2\alpha_6\alpha_4+\alpha_5\alpha_3\alpha_4+\alpha_5\alpha_6\alpha_4+\alpha_3\alpha_6\alpha_4)W_2 \\
&\quad +(\alpha_2\alpha_5\alpha_3\alpha_6+\alpha_2\alpha_5\alpha_3\alpha_4+\alpha_2\alpha_5\alpha_6\alpha_4+\alpha_2\alpha_3\alpha_6\alpha_4+\alpha_5\alpha_3\alpha_6\alpha_4)W_1 - \alpha_2\alpha_5\alpha_3\alpha_6\alpha_4W_0, \\
b_2 &= W_5 - (\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6)W_4 + (\alpha_1\alpha_5+\alpha_1\alpha_3+\alpha_1\alpha_6+\alpha_1\alpha_4+\alpha_5\alpha_3+\alpha_5\alpha_6+\alpha_5\alpha_4+\alpha_3\alpha_6+\alpha_3\alpha_4+\alpha_6\alpha_4)W_3 \\
&\quad -(\alpha_1\alpha_5\alpha_3+\alpha_1\alpha_5\alpha_6+\alpha_1\alpha_5\alpha_4+\alpha_1\alpha_3\alpha_6+\alpha_1\alpha_3\alpha_4+\alpha_1\alpha_6\alpha_4+\alpha_5\alpha_3\alpha_6+\alpha_5\alpha_3\alpha_4+\alpha_5\alpha_6\alpha_4+\alpha_3\alpha_6\alpha_4)W_2 \\
&\quad +(\alpha_1\alpha_5\alpha_3\alpha_6+\alpha_1\alpha_5\alpha_3\alpha_4+\alpha_1\alpha_5\alpha_6\alpha_4+\alpha_1\alpha_3\alpha_6\alpha_4+\alpha_5\alpha_3\alpha_6\alpha_4)W_1 - \alpha_1\alpha_5\alpha_3\alpha_6\alpha_4W_0, \\
b_3 &= W_5 - (\alpha_1+\alpha_2+\alpha_4+\alpha_5+\alpha_6)W_4 + (\alpha_1\alpha_2+\alpha_1\alpha_5+\alpha_1\alpha_6+\alpha_2\alpha_5+\alpha_1\alpha_4+\alpha_2\alpha_6+\alpha_5\alpha_6+\alpha_2\alpha_4+\alpha_5\alpha_4+\alpha_6\alpha_4)W_3 \\
&\quad -(\alpha_1\alpha_2\alpha_5+\alpha_1\alpha_2\alpha_6+\alpha_1\alpha_5\alpha_6+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_5\alpha_4+\alpha_2\alpha_5\alpha_6+\alpha_1\alpha_6\alpha_4+\alpha_2\alpha_5\alpha_4+\alpha_2\alpha_6\alpha_4+\alpha_5\alpha_6\alpha_4)W_2 \\
&\quad +(\alpha_1\alpha_2\alpha_5\alpha_6+\alpha_1\alpha_2\alpha_5\alpha_4+\alpha_1\alpha_2\alpha_6\alpha_4+\alpha_1\alpha_5\alpha_6\alpha_4+\alpha_2\alpha_5\alpha_6\alpha_4)W_1 - \alpha_1\alpha_2\alpha_5\alpha_6\alpha_4W_0, \\
b_4 &= W_5 - (\alpha_1+\alpha_2+\alpha_3+\alpha_5+\alpha_6)W_4 + (\alpha_1\alpha_2+\alpha_1\alpha_5+\alpha_1\alpha_3+\alpha_1\alpha_6+\alpha_2\alpha_5+\alpha_2\alpha_3+\alpha_2\alpha_6+\alpha_5\alpha_3+\alpha_5\alpha_6+\alpha_3\alpha_6)W_3 \\
&\quad -(\alpha_1\alpha_2\alpha_5+\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_6+\alpha_1\alpha_5\alpha_3+\alpha_1\alpha_5\alpha_6+\alpha_1\alpha_3\alpha_6+\alpha_2\alpha_5\alpha_3+\alpha_2\alpha_5\alpha_6+\alpha_2\alpha_3\alpha_6+\alpha_5\alpha_3\alpha_6)W_2 \\
&\quad +(\alpha_1\alpha_2\alpha_5\alpha_3+\alpha_1\alpha_2\alpha_5\alpha_6+\alpha_1\alpha_2\alpha_3\alpha_6+\alpha_1\alpha_5\alpha_3\alpha_6+\alpha_2\alpha_5\alpha_3\alpha_6)W_1 - \alpha_1\alpha_2\alpha_5\alpha_3\alpha_6W_0, \\
b_5 &= W_5 - (\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_6)W_4 + (\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_6+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_2\alpha_6+\alpha_2\alpha_4+\alpha_3\alpha_6+\alpha_3\alpha_4+\alpha_6\alpha_4)W_3 \\
&\quad -(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_6+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_6+\alpha_1\alpha_3\alpha_4+\alpha_1\alpha_6\alpha_4+\alpha_2\alpha_3\alpha_6+\alpha_2\alpha_3\alpha_4+\alpha_2\alpha_6\alpha_4+\alpha_3\alpha_6\alpha_4)W_2 \\
&\quad +(\alpha_1\alpha_2\alpha_3\alpha_6+\alpha_1\alpha_2\alpha_3\alpha_4+\alpha_1\alpha_2\alpha_6\alpha_4+\alpha_1\alpha_3\alpha_6\alpha_4+\alpha_2\alpha_3\alpha_6\alpha_4)W_1 - \alpha_1\alpha_2\alpha_3\alpha_6\alpha_4W_0, \\
b_6 &= W_5 - (\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)W_4 + (\alpha_1\alpha_2+\alpha_1\alpha_5+\alpha_1\alpha_3+\alpha_2\alpha_5+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_5\alpha_3+\alpha_2\alpha_4+\alpha_5\alpha_4+\alpha_3\alpha_4)W_3 \\
&\quad -(\alpha_1\alpha_2\alpha_5+\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_5\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_5\alpha_4+\alpha_2\alpha_5\alpha_3+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_5\alpha_4+\alpha_2\alpha_3\alpha_4+\alpha_5\alpha_3\alpha_4)W_2 \\
&\quad +(\alpha_1\alpha_2\alpha_5\alpha_3+\alpha_1\alpha_2\alpha_5\alpha_4+\alpha_1\alpha_2\alpha_3\alpha_4+\alpha_1\alpha_5\alpha_3\alpha_4+\alpha_2\alpha_5\alpha_3\alpha_4)W_1 - \alpha_1\alpha_2\alpha_5\alpha_3\alpha_4W_0.
\end{aligned}$$

Usually, it is customary to choose $r_1, r_2, r_3, r_4, r_5, r_6$ so that the Equ. (1.2) has at least one real (say $\alpha_1$) solutions.

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers $n$, for a proof of this result see [1]. This result of Howard and Saidak [1] is even true in the case of higher-order recurrence relations.

In this paper we consider the case $r_1 = 2$, $r_2 = 3, r_3 = 5, r_4 = 7, r_5 = 11, r_6 = 13$ and in this case we write $V_n = W_n$. A generalized 6-primes sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4, V_5)\}_{n\geq 0}$ is defined by the sixth-order recurrence relations

$$V_n = 2V_{n-1} + 3V_{n-2} + 5V_{n-3} + 7V_{n-4} + 11V_{n-5} + 13V_{n-6} \tag{1.4}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3, V_4 = c_4, V_5 = c_5$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{11}{13}V_{-(n-1)} - \frac{7}{13}V_{-(n-2)} - \frac{5}{13}V_{-(n-3)} - \frac{3}{13}V_{-(n-4)} - \frac{2}{13}V_{-(n-5)} + \frac{1}{13}V_{-(n-6)}$$

for $n = 1, 2, 3, ....$Therefore, recurrence (1.4) holds for all integer $n$.

(1.3) can be used to obtain Binet formula of generalized 6-primes numbers. Binet formula of generalized 6-primes numbers can be given as

$$V_n = \sum_{k=1}^{6} \frac{b_k \theta_k^n}{\prod\limits_{\substack{j=1 \\ k\neq j}}^{6} (\theta_k - \theta_j)} \tag{1.5}$$

where

$$\begin{aligned}
b_1 &= V_5 - (\theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6)V_4 + (\theta_2\theta_5 + \theta_2\theta_3 + \theta_2\theta_6 + \theta_5\theta_3 + \theta_5\theta_6 + \theta_2\theta_4 + \theta_5\theta_4 + \theta_3\theta_6 + \theta_3\theta_4 + \theta_6\theta_4)V_3 \\
&\quad -(\theta_2\theta_5\theta_3 + \theta_2\theta_5\theta_6 + \theta_2\theta_5\theta_4 + \theta_2\theta_3\theta_6 + \theta_5\theta_3\theta_6 + \theta_2\theta_3\theta_4 + \theta_2\theta_6\theta_4 + \theta_5\theta_3\theta_4 + \theta_5\theta_6\theta_4 + \theta_3\theta_6\theta_4)V_2 \\
&\quad +(\theta_2\theta_5\theta_3\theta_6 + \theta_2\theta_5\theta_3\theta_4 + \theta_2\theta_5\theta_6\theta_4 + \theta_2\theta_3\theta_6\theta_4 + \theta_5\theta_3\theta_6\theta_4)V_1 - \theta_2\theta_5\theta_3\theta_6\theta_4 V_0, \\
b_2 &= V_5 - (\theta_1 + \theta_3 + \theta_4 + \theta_5 + \theta_6)V_4 + (\theta_1\theta_5 + \theta_1\theta_3 + \theta_1\theta_6 + \theta_1\theta_4 + \theta_5\theta_3 + \theta_5\theta_6 + \theta_5\theta_4 + \theta_3\theta_6 + \theta_3\theta_4 + \theta_6\theta_4)V_3 \\
&\quad -(\theta_1\theta_5\theta_3 + \theta_1\theta_5\theta_6 + \theta_1\theta_5\theta_4 + \theta_1\theta_3\theta_6 + \theta_1\theta_3\theta_4 + \theta_1\theta_6\theta_4 + \theta_5\theta_3\theta_6 + \theta_5\theta_3\theta_4 + \theta_5\theta_6\theta_4 + \theta_3\theta_6\theta_4)V_2 \\
&\quad +(\theta_1\theta_5\theta_3\theta_6 + \theta_1\theta_5\theta_3\theta_4 + \theta_1\theta_5\theta_6\theta_4 + \theta_1\theta_3\theta_6\theta_4 + \theta_5\theta_3\theta_6\theta_4)V_1 - \theta_1\theta_5\theta_3\theta_6\theta_4 V_0, \\
b_3 &= V_5 - (\theta_1 + \theta_2 + \theta_4 + \theta_5 + \theta_6)V_4 + (\theta_1\theta_2 + \theta_1\theta_5 + \theta_1\theta_6 + \theta_2\theta_5 + \theta_1\theta_4 + \theta_2\theta_6 + \theta_5\theta_6 + \theta_2\theta_4 + \theta_5\theta_4 + \theta_6\theta_4)V_3 \\
&\quad -(\theta_1\theta_2\theta_5 + \theta_1\theta_2\theta_6 + \theta_1\theta_5\theta_6 + \theta_1\theta_2\theta_4 + \theta_1\theta_5\theta_4 + \theta_2\theta_5\theta_6 + \theta_1\theta_6\theta_4 + \theta_2\theta_5\theta_4 + \theta_2\theta_6\theta_4 + \theta_5\theta_6\theta_4)V_2 \\
&\quad +(\theta_1\theta_2\theta_5\theta_6 + \theta_1\theta_2\theta_5\theta_4 + \theta_1\theta_2\theta_6\theta_4 + \theta_1\theta_5\theta_6\theta_4 + \theta_2\theta_5\theta_6\theta_4)V_1 - \theta_1\theta_2\theta_5\theta_6\theta_4 V_0, \\
b_4 &= V_5 - (\theta_1 + \theta_2 + \theta_3 + \theta_5 + \theta_6)V_4 + (\theta_1\theta_2 + \theta_1\theta_5 + \theta_1\theta_3 + \theta_1\theta_6 + \theta_2\theta_5 + \theta_2\theta_3 + \theta_2\theta_6 + \theta_5\theta_3 + \theta_5\theta_6 + \theta_3\theta_6)V_3 \\
&\quad -(\theta_1\theta_2\theta_5 + \theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_6 + \theta_1\theta_5\theta_3 + \theta_1\theta_5\theta_6 + \theta_1\theta_3\theta_6 + \theta_2\theta_5\theta_3 + \theta_2\theta_5\theta_6 + \theta_2\theta_3\theta_6 + \theta_5\theta_3\theta_6)V_2 \\
&\quad +(\theta_1\theta_2\theta_5\theta_3 + \theta_1\theta_2\theta_5\theta_6 + \theta_1\theta_2\theta_3\theta_6 + \theta_1\theta_5\theta_3\theta_6 + \theta_2\theta_5\theta_3\theta_6)V_1 - \theta_1\theta_2\theta_5\theta_3\theta_6 V_0, \\
b_5 &= V_5 - (\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_6)V_4 + (\theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_6 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_2\theta_6 + \theta_2\theta_4 + \theta_3\theta_6 + \theta_3\theta_4 + \theta_6\theta_4)V_3 \\
&\quad -(\theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_6 + \theta_1\theta_2\theta_4 + \theta_1\theta_3\theta_6 + \theta_1\theta_3\theta_4 + \theta_1\theta_6\theta_4 + \theta_2\theta_3\theta_6 + \theta_2\theta_3\theta_4 + \theta_2\theta_6\theta_4 + \theta_3\theta_6\theta_4)V_2 \\
&\quad +(\theta_1\theta_2\theta_3\theta_6 + \theta_1\theta_2\theta_3\theta_4 + \theta_1\theta_2\theta_6\theta_4 + \theta_1\theta_3\theta_6\theta_4 + \theta_2\theta_3\theta_6\theta_4)V_1 - \theta_1\theta_2\theta_3\theta_6\theta_4 V_0, \\
b_6 &= V_5 - (\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5)V_4 + (\theta_1\theta_2 + \theta_1\theta_5 + \theta_1\theta_3 + \theta_2\theta_5 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_5\theta_3 + \theta_2\theta_4 + \theta_5\theta_4 + \theta_3\theta_4)V_3 \\
&\quad -(\theta_1\theta_2\theta_5 + \theta_1\theta_2\theta_3 + \theta_1\theta_5\theta_3 + \theta_1\theta_2\theta_4 + \theta_1\theta_5\theta_4 + \theta_2\theta_5\theta_3 + \theta_1\theta_3\theta_4 + \theta_2\theta_5\theta_4 + \theta_2\theta_3\theta_4 + \theta_5\theta_3\theta_4)V_2 \\
&\quad +(\theta_1\theta_2\theta_5\theta_3 + \theta_1\theta_2\theta_5\theta_4 + \theta_1\theta_2\theta_3\theta_4 + \theta_1\theta_5\theta_3\theta_4 + \theta_2\theta_5\theta_3\theta_4)V_1 - \theta_1\theta_2\theta_5\theta_3\theta_4 V_0.
\end{aligned}$$

Here, $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and $\theta_6$ are the roots of the equation

$$x^6 - 2x^5 - 3x^4 - 5x^3 - 7x^2 - 11x - 13 = 0. \tag{1.6}$$

Moreover, the approximate value of the roots $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and $\theta_6$ of Equation (1.6) are given by

$$\begin{aligned}
\theta_1 &= 3.515372711921757, \\
\theta_2 &= -1.183212731145181, \\
\theta_3 &= -0.7228110394202282 + 1.063369120765496i, \\
\theta_4 &= -0.7228110394202282 - 1.063369120765496i, \\
\theta_5 &= 0.5567310490319399 + 1.257207305141223i, \\
\theta_6 &= 0.5567310490319399 - 1.257207305141223i.
\end{aligned}$$

The first few generalized 6-primes numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized 6-primes numbers

| $n$ | $V_n$ | $V_{-n}$ |
|---|---|---|
| 0 | $V_0$ | |
| 1 | $V_1$ | $\frac{1}{13}V_5 - \frac{7}{13}V_1 - \frac{5}{13}V_2 - \frac{3}{13}V_3 - \frac{2}{13}V_4 - \frac{11}{13}V_0$ |
| 2 | $V_2$ | $\frac{30}{169}V_0 + \frac{12}{169}V_1 + \frac{16}{169}V_2 + \frac{7}{169}V_3 + \frac{35}{169}V_4 - \frac{11}{169}V_5$ |
| 3 | $V_3$ | $\frac{365}{2197}V_3 - \frac{2}{2197}V_1 - \frac{59}{2197}V_2 - \frac{174}{2197}V_0 - \frac{203}{2197}V_4 + \frac{30}{2197}V_5$ |
| 4 | $V_4$ | $\frac{1888}{28\,561}V_0 + \frac{451}{28\,561}V_1 + \frac{5615}{28\,561}V_2 - \frac{2117}{28\,561}V_3 + \frac{738}{28\,561}V_4 - \frac{174}{28\,561}V_5$ |
| 5 | $V_5$ | $\frac{59\,779}{371\,293}V_1 - \frac{14\,905}{371\,293}V_0 - \frac{36\,961}{371\,293}V_2 + \frac{3930}{371\,293}V_3 - \frac{6038}{371\,293}V_4 + \frac{1888}{371\,293}V_5$ |
| 6 | $13V_0 + 11V_1 + 7V_2 + 5V_3 + 3V_4 + 2V_5$ | $\frac{941\,082}{4826\,809}V_0 - \frac{376\,158}{4826\,809}V_1 + \frac{125\,615}{4826\,809}V_2 - \frac{33\,779}{4826\,809}V_3 + \frac{54\,354}{4826\,809}V_4 - \frac{14\,905}{4826\,809}V_5$ |
| 7 | $26V_0 + 35V_1 + 25V_2 + 17V_3 + 11V_4 + 7V_5$ | $\frac{941\,082}{62\,748\,517}V_5 - \frac{4954\,579}{62\,748\,517}V_1 - \frac{5144\,537}{62\,748\,517}V_2 - \frac{2116\,644}{62\,748\,517}V_3 - \frac{2075\,929}{62\,748\,517}V_4 - \frac{15\,241\,956}{62\,748\,517}V_0$ |

Now we define three special cases of the sequence $\{V_n\}$. 6-primes sequence$\{G_n\}_{n\geq 0}$, Lucas 6-primes sequence $\{H_n\}_{n\geq 0}$ and modified 6-primes sequence $\{E_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_{n+6} = 2G_{n+5} + 3G_{n+4} + 5G_{n+3} + 7G_{n+2} + 11G_{n+1} + 13G_n, \qquad G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 2,$$

$$H_{n+6} = 2H_{n+5} + 3H_{n+4} + 5H_{n+3} + 7H_{n+2} + 11H_{n+1} + 13H_n, \qquad H_0 = 6, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542, \tag{1.7}$$

and

$$E_{n+6} = 2E_{n+5} + 3E_{n+4} + 5E_{n+3} + 7E_{n+2} + 11E_{n+1} + 13E_n, \qquad E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1, \tag{1.8}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{11}{13}G_{-(n-1)} - \frac{7}{13}G_{-(n-2)} - \frac{5}{13}G_{-(n-3)} - \frac{3}{13}G_{-(n-4)} - \frac{2}{13}G_{-(n-5)} + \frac{1}{13}G_{-(n-6)}, \tag{1.9}$$

$$H_{-n} = -\frac{11}{13}H_{-(n-1)} - \frac{7}{13}H_{-(n-2)} - \frac{5}{13}H_{-(n-3)} - \frac{3}{13}H_{-(n-4)} - \frac{2}{13}H_{-(n-5)} + \frac{1}{13}H_{-(n-6)} \tag{1.10}$$

and

$$E_{-n} = -\frac{11}{13}E_{-(n-1)} - \frac{7}{13}E_{-(n-2)} - \frac{5}{13}E_{-(n-3)} - \frac{3}{13}E_{-(n-4)} - \frac{2}{13}E_{-(n-5)} + \frac{1}{13}E_{-(n-6)} \tag{1.11}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.9), (1.10) and (1.11) hold for all integer $n$.

Note that the sequences $G_n, H_n$ and $E_n$ are not indexed in [5] yet. Next, we present the first few values of the 6-primes, Lucas 6-primes and modified 6-primes numbers with positive and negative subscripts:

Table 2. The first few values of the special sisth-order numbers with positive and negative subscripts.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $G_n$ | 0 | 0 | 0 | 0 | 1 | 2 | 7 | 25 | 88 | 311 |
| $G_{-n}$ | .... | 0 | $\frac{1}{13}$ | $-\frac{11}{169}$ | $\frac{30}{2197}$ | $-\frac{174}{28561}$ | $\frac{1888}{371293}$ | $-\frac{14905}{4826809}$ | $\frac{941082}{62748517}$ | $-\frac{15241956}{815730721}$ |
| $H_n$ | 6 | 2 | 10 | 41 | 150 | 542 | 1909 | 6617 | 23302 | 81977 |
| $H_{-n}$ | .... | $-\frac{11}{13}$ | $-\frac{61}{169}$ | $-\frac{863}{2197}$ | $-\frac{2025}{28561}$ | $-\frac{60756}{371293}$ | $\frac{4839977}{4826809}$ | $-\frac{40911574}{62748517}$ | $\frac{100922415}{815730721}$ | $-\frac{1281284909}{10604499373}$ |
| $E_n$ | 0 | 0 | 0 | 0 | 1 | 1 | 5 | 18 | 63 | 223 |
| $E_{-n}$ | .... | $-\frac{1}{13}$ | $\frac{24}{169}$ | $-\frac{173}{2197}$ | $\frac{564}{28561}$ | $-\frac{4150}{371293}$ | $\frac{39449}{4826809}$ | $-\frac{1134847}{62748517}$ | $\frac{27476022}{815730721}$ | $-\frac{301397417}{10604499373}$ |

For all integers $n$, 6-primes, Lucas 6-primes and modified 6-primes numbers (using initial conditions in (1.5)) can be expressed using Binet's formulas as

$$
\begin{aligned}
G_n &= \sum_{k=1}^{6} \frac{\theta_k^{n+1}}{\prod\limits_{\substack{j=1 \\ k\neq j}}^{6} (\theta_k - \theta_j)}, \\
H_n &= \sum_{k=1}^{6} \theta_k^n = \theta_1^n + \theta_2^n + \theta_3^n + \theta_4^n + \theta_5^n + \theta_6^n, \\
E_n &= \sum_{k=1}^{6} \frac{(\theta_k - 1)\theta_k^{n+1}}{\prod\limits_{\substack{j=1 \\ k\neq j}}^{6} (\theta_k - \theta_j)},
\end{aligned}
$$

respectively.

## 2. Generating Functions

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence $V_n$.

Lemma 2. *Suppose that $f_{V_n}(x) = \sum\limits_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized 6-primes sequence $\{V_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{\Lambda}{1 - 2x - 3x^2 - 5x^3 - 7x^4 - 11x^5 - 13x^6}. \tag{2.1}$$

*where*

$$
\begin{aligned}
\Lambda &= V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2 + (V_3 - 2V_2 - 3V_1 - 5V_0)x^3 \\
&\quad + (V_4 - 2V_3 - 3V_2 - 5V_1 - 7V_0)x^4 + (V_5 - 2V_4 - 3V_3 - 5V_2 - 7V_1 - 11V_0)x^5 \\
&= V_0 + \sum_{i=1}^{6-1} x^i \left( V_i - \sum_{j=1}^{i} r_j V_{i-j} \right)
\end{aligned}
$$

Proof. Using the definition of generalized 6-primes numbers, and substracting $2x\sum_{n=0}^{\infty}V_nx^n$, $3x^2\sum_{n=0}^{\infty}V_nx^n$, $5x^3\sum_{n=0}^{\infty}V_nx^n$, $7x^4\sum_{n=0}^{\infty}V_nx^n$, $11x^5\sum_{n=0}^{\infty}V_nx^n$ and $13x^6\sum_{n=0}^{\infty}V_nx^n$from $\sum_{n=0}^{\infty}V_nx^n$ we obtain

$$
\begin{aligned}
&(1-2x-3x^2-5x^3-7x^4-11x^5-13x^6)\sum_{n=0}^{\infty}V_nx^n \\
=\;& \sum_{n=0}^{\infty}V_nx^n-2x\sum_{n=0}^{\infty}V_nx^n-3x^2\sum_{n=0}^{\infty}V_nx^n-5x^3\sum_{n=0}^{\infty}V_nx^n-7x^4\sum_{n=0}^{\infty}V_nx^n \\
&-11x^5\sum_{n=0}^{\infty}V_nx^n-13x^6\sum_{n=0}^{\infty}V_nx^n \\
=\;& \sum_{n=0}^{\infty}V_nx^n-2\sum_{n=0}^{\infty}V_nx^{n+1}-3\sum_{n=0}^{\infty}V_nx^{n+2}-5\sum_{n=0}^{\infty}V_nx^{n+3}-7\sum_{n=0}^{\infty}V_nx^{n+4} \\
&-11\sum_{n=0}^{\infty}V_nx^{n+5}-13\sum_{n=0}^{\infty}V_nx^{n+6} \\
=\;& \sum_{n=0}^{\infty}V_nx^n-2\sum_{n=1}^{\infty}V_{n-1}x^n-3\sum_{n=2}^{\infty}V_{n-2}x^n-5\sum_{n=3}^{\infty}V_{n-3}x^n-7\sum_{n=4}^{\infty}V_{n-4}x^n \\
&-11\sum_{n=5}^{\infty}V_{n-5}x^n-13\sum_{n=6}^{\infty}V_{n-6}x^n \\
=\;& (V_0+V_1x+V_2x^2+V_3x^3+V_4x^4+V_5x^5)-2(V_0x+V_1x^2+V_2x^3+V_3x^4+V_4x^5) \\
&-3(V_0x^2+V_1x^3+V_2x^4+V_3x^5)-5(V_0x^3+V_1x^4+V_2x^5)-7(V_0x^4+V_1x^5)-11V_0x^5 \\
&+\sum_{n=6}^{\infty}(V_n-2V_{n-1}-3V_{n-2}-5V_{n-3}-7V_{n-4}-11V_{n-5}-13V_{n-6})x^n \\
=\;& V_0+(V_1-2V_0)x+(V_2-2V_1-3V_0)x^2+(V_3-2V_2-3V_1-5V_0)x^3 \\
&+(V_4-2V_3-3V_2-5V_1-7V_0)x^4+(V_5-2V_4-3V_3-5V_2-7V_1-11V_0)x^5 \\
=\;& V_0+\sum_{i=1}^{6-1}x^i\left(V_i-\sum_{j=1}^{i}r_jV_{i-j}\right).
\end{aligned}
$$

Rearranging above equation, we obtain (2.1).

The previous lemma gives the following results as particular examples.

Corollary 3. *Generated functions of 6-primes, Lucas 6-primes and modified 6-primes numbers are*

$$\sum_{n=0}^{\infty}H_nx^n=\frac{x^4}{1-2x-3x^2-5x^3-7x^4-11x^5-13x^6},$$

*is true is true : and*

$$\sum_{n=0}^{\infty}H_nx^n=\frac{6-10x-12x^2-15x^3-14x^4-11x^5}{1-2x-3x^2-5x^3-7x^4-11x^5-13x^6},$$

*is true is true and*

$$\sum_{n=0}^{\infty}E_nx^n=\frac{x^4-x^5}{1-2x-3x^2-5x^3-7x^4-11x^5-13x^6},$$

*respectively.*

### 3. Obtaining Binet Formula From Generating Function

We next find Binet formula of generalized 6-primes numbers $\{V_n\}$ by the use of generating function for $V_n$.

THEOREM 4. *(Binet formula of generalized 6-primes numbers)*

$$V_n = \sum_{k=1}^{6} \frac{d_k \theta_k^n}{\prod\limits_{\substack{j=1\\k\neq j}}^{6} (\theta_k - \theta_j)} \tag{3.1}$$

*where*

$$\begin{aligned} d_1 &= V_0\theta_1^{6-1} + \sum_{i=1}^{6-1} \theta_1^{6-1-i} \left[ V_i - \sum_{j=1}^{i} r_j V_{i-j} \right], \\ d_l &= V_0\theta_l^{6-1} + \sum_{i=1}^{6-1} \theta_l^{6-1-i} \left[ V_i - \sum_{j=1}^{i} r_j V_{i-j} \right], \quad 1 \le l \le m = 6, \\ r_1 &= 2, r_2 = 3, r_3 = 5, r_4 = 7, r_5 = 11, r_6 = 13. \end{aligned}$$

*Proof.* Let

$$h(x) = 1 - 2x - 3x^2 - 5x^3 - 7x^4 - 11x^5 - 13x^6.$$

Then for some $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and $\theta_6$ we write

$$h(x) = (1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x)$$

i.e.,

$$1 - 2x - 3x^2 - 5x^3 - 7x^4 - 11x^5 - 13x^6 = (1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x) \tag{3.2}$$

Hence $\frac{1}{\theta_1}, \frac{1}{\theta_2}, \frac{1}{\theta_3}, \frac{1}{\theta_4}, \frac{1}{\theta_5}$ and $\frac{1}{\theta_6}$ are the roots of $h(x)$. This gives $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and $\theta_6$ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{2}{x} - \frac{3}{x^2} - \frac{5}{x^3} - \frac{7}{x^4} - \frac{11}{x^5} - \frac{13}{x^6} = 0.$$

This implies $x^6 - 2x^5 - 3x^4 - 5x^3 - 7x^2 - 11x - 13 = 0$. Now, by (2.1) and (3.2), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + \sum_{i=1}^{6-1} x^i \left[ V_i - \sum_{j=1}^{i} r_j V_{i-j} \right]}{(1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x)}.$$

Then we write

$$\frac{V_0 + \sum_{i=1}^{6-1} x^i \left[ V_i - \sum_{j=1}^{i} r_j V_{i-j} \right]}{(1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x)} = \frac{A_1}{(1-\theta_1 x)} + \frac{A_2}{(1-\theta_2 x)} + \frac{A_3}{(1-\theta_3 x)} + \frac{A_4}{(1-\theta_4 x)} + \frac{A_5}{(1-\theta_5 x)} + \frac{A_6}{(1-\theta_6 x)}. \tag{3.3}$$

So

$$\begin{aligned} & V_0 + \sum_{i=1}^{6-1} x^i \left[ V_i - \sum_{j=1}^{i} r_j V_{i-j} \right] \\ = \; & A_1(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x) + A_2(1-\theta_1 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x) \\ & + A_3(1-\theta_1 x)(1-\theta_2 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x) + A_4(1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_5 x)(1-\theta_6 x) \\ & + A_5(1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_6 x) + A_6(1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x). \end{aligned}$$

If we consider $x = \frac{1}{\theta_1}$, we get

$$V_0 + \sum_{i=1}^{6-1} \left(\frac{1}{\theta_1}\right)^i \left[ V_i - \sum_{j=1}^{i} r_j V_{i-j} \right] = A_1(1 - \frac{\theta_2}{\theta_1})(1 - \frac{\theta_3}{\theta_1})(1 - \frac{\theta_4}{\theta_1})(1 - \frac{\theta_5}{\theta_1})(1 - \frac{\theta_6}{\theta_1}).$$

This gives

$$
\begin{aligned}
A_1 &= \frac{\theta_1^5(V_0+\sum_{i=1}^{6-1}\left(\frac{1}{\theta_1}\right)^i\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right])}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_1-\theta_5)(\theta_1-\theta_6)} \\
&= \frac{V_0\theta_1^{6-1}+\sum_{i=1}^{6-1}\theta_1^{6-1-i}\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right]}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_1-\theta_5)(\theta_1-\theta_6)}
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
A_2 &= \frac{V_0\theta_2^{6-1}+\sum_{i=1}^{6-1}\theta_2^{6-1-i}\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right]}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)(\theta_2-\theta_5)(\theta_2-\theta_6)}, \\
A_3 &= \frac{V_0\theta_3^{6-1}+\sum_{i=1}^{6-1}\theta_3^{6-1-i}\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right]}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)(\theta_3-\theta_5)(\theta_3-\theta_6)}, \\
A_4 &= \frac{V_0\theta_4^{6-1}+\sum_{i=1}^{6-1}\theta_4^{6-1-i}\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right]}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)(\theta_4-\theta_5)(\theta_4-\theta_6)}, \\
A_5 &= \frac{V_0\theta_5^{6-1}+\sum_{i=1}^{6-1}\theta_5^{6-1-i}\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right]}{(\theta_5-\theta_1)(\theta_5-\theta_2)(\theta_5-\theta_3)(\theta_5-\theta_4)(\theta_5-\theta_6)}, \\
A_6 &= \frac{V_0\theta_6^{6-1}+\sum_{i=1}^{6-1}\theta_6^{6-1-i}\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right]}{(\theta_6-\theta_1)(\theta_6-\theta_2)(\theta_6-\theta_3)(\theta_6-\theta_4)(\theta_6-\theta_5)}.
\end{aligned}
$$

Thus (3.3) can be written as

$$
\sum_{n=0}^{\infty}V_nx^n = A_1(1-\theta_1x)^{-1}+A_2(1-\theta_2x)^{-1}+A_3(1-\theta_3x)^{-1}+A_4(1-\theta_4x)^{-1}+A_5(1-\theta_5x)^{-1}+A_6(1-\theta_6x)^{-1}.
$$

This gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}V_nx^n &= A_1\sum_{n=0}^{\infty}\theta_1^nx^n+A_2\sum_{n=0}^{\infty}\theta_2^nx^n+A_3\sum_{n=0}^{\infty}\theta_3^nx^n+A_4\sum_{n=0}^{\infty}\theta_4^nx^n+A_5\sum_{n=0}^{\infty}\theta_5^nx^n+A_6\sum_{n=0}^{\infty}\theta_6^nx^n \\
&= \sum_{n=0}^{\infty}(A_1\theta_1^n+A_2\theta_2^n+A_3\theta_3^n+A_4\theta_4^n+A_5\theta_5^n+A_6\theta_6^n)x^n.
\end{aligned}
$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$
V_n = A_1\theta_1^n+A_2\theta_2^n+A_3\theta_3^n+A_4\theta_4^n+A_5\theta_5^n+A_6\theta_6^n
$$

and then we get (3.1).

Next, using Theorem 4, we present the Binet formulas of 6-primes, Lucas 6-primes and modified 6-primes sequences.

Corollary 5. *Binet formulas of 6-primes, Lucas 6-primes and modified 6-primes sequences are*

$$
\begin{aligned}
G_n &= \sum_{k=1}^{6}\frac{\theta_k^{n+1}}{\prod\limits_{\substack{j=1\\k\neq j}}^{6}(\theta_k-\theta_j)}, \\
H_n &= \sum_{k=1}^{6}\theta_k^n = \theta_1^n+\theta_2^n+\theta_3^n+\theta_4^n+\theta_5^n+\theta_6^n, \\
E_n &= \sum_{k=1}^{6}\frac{(\theta_k-1)\theta_k^{n+1}}{\prod\limits_{\substack{j=1\\k\neq j}}^{6}(\theta_k-\theta_j)},
\end{aligned}
$$

*respectively.*

## 4. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized 6-primes sequence $\{V_n\}_{n\geq 0}$.

THEOREM 6 (Simson Formula of Generalized 6-primes Numbers). *For all integers $n$, we have*

$$\begin{vmatrix} V_{n+5} & V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} & V_{n-5} \end{vmatrix} = (-1)^n 13^n \begin{vmatrix} V_5 & V_4 & V_3 & V_2 & V_1 & V_0 \\ V_4 & V_3 & V_2 & V_1 & V_0 & V_{-1} \\ V_3 & V_2 & V_1 & V_0 & V_{-1} & V_{-2} \\ V_2 & V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} \\ V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} \\ V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} & V_{-5} \end{vmatrix}.$$

*Proof.* It is given in Soykan [6].

The previous theorem gives the following results as particular examples.

COROLLARY 7. *For all integers $n$, Simson formula of 6-primes, Lucas 6-primes and modified 6-primes numbers are given as*

$$\begin{vmatrix} G_{n+5} & G_{n+4} & G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+4} & G_{n+3} & G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+3} & G_{n+2} & G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} & G_{n-2} & G_{n-3} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} & G_{n-3} & G_{n-4} \\ G_n & G_{n-1} & G_{n-2} & G_{n-3} & G_{n-4} & G_{n-5} \end{vmatrix} = (-1)^{n+1} 13^{n-4} \quad (4.1)$$

*and*

$$\begin{vmatrix} H_{n+5} & H_{n+4} & H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+4} & H_{n+3} & H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+3} & H_{n+2} & H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} & H_{n-2} & H_{n-3} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} & H_{n-3} & H_{n-4} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} & H_{n-4} & H_{n-5} \end{vmatrix} = 99191747 \times 2^5 \times 41 \times (-1)^{n+1} 13^{n-5} \quad (4.2)$$

*and*

$$\begin{vmatrix} E_{n+5} & E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_{n} \\ E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_{n} & E_{n-1} \\ E_{n+3} & E_{n+2} & E_{n+1} & E_{n} & E_{n-1} & E_{n-2} \\ E_{n+2} & E_{n+1} & E_{n} & E_{n-1} & E_{n-2} & E_{n-3} \\ E_{n+1} & E_{n} & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} \\ E_{n} & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} & E_{n-5} \end{vmatrix} = 5 \times 2^{3} \times (-1)^{n+1} 13^{n-5} \tag{4.3}$$

*respectively.*

## 5. Some Identities

In this section, we obtain some identities of 6-primes, Lucas 6-primes and modified 6-primes numbers. First, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

LEMMA 8. *The following equalities are true:*

$$\begin{aligned} 169H_n &= -61G_{n+6} - 21G_{n+5} + 1483G_{n+4} - 956G_{n+3} - 886G_{n+2} - 863G_{n+1} \qquad (5.1) \\ 13H_n &= -11G_{n+5} + 100G_{n+4} - 97G_{n+3} - 101G_{n+2} - 118G_{n+1} - 61G_n \\ H_n &= 6G_{n+4} - 10G_{n+3} - 12G_{n+2} - 15G_{n+1} - 14G_n - 11G_{n-1} \\ H_n &= 2G_{n+3} + 6G_{n+2} + 15G_{n+1} + 28G_n + 55G_{n-1} + 78G_{n-2} \\ H_n &= 10G_{n+2} + 21G_{n+1} + 38G_n + 69G_{n-1} + 100G_{n-2} + 26G_{n-3} \\ H_n &= 41G_{n+1} + 68G_n + 119G_{n-1} + 170G_{n-2} + 136G_{n-3} + 130G_{n-4} \end{aligned}$$

*and*

$$\begin{aligned} 65069786032G_n &= 17165493H_{n+6} - 48224301H_{n+5} + 84682036H_{n+4} - 663056677H_{n+3} \\ &\quad +802372816H_{n+2} - 86032399H_{n+1}, \\ 65069786032G_n &= -13893315H_{n+5} + 136178515H_{n+4} - 577229212H_{n+3} + 922531267H_{n+2} \\ &\quad +102788024H_{n+1} + 223151409H_n, \\ 65069786032G_n &= 108391885H_{n+4} - 618909157H_{n+3} + 853064692H_{n+2} + 5534819H_{n+1} \\ &\quad +70324944H_n - 180613095H_{n-1}, \\ 65069786032G_n &= -402125387H_{n+3} + 1178240347H_{n+2} + 547494244H_{n+1} + 829068139H_n \\ &\quad +1011697640H_{n-1} + 1409094505H_{n-2}, \\ 65069786032G_n &= 373989573H_{n+2} - 658881917H_{n+1} - 1181558796H_n - 1803180069H_{n-1} \\ &\quad -3014284752H_{n-2} - 5227630031H_{n-3}, \\ 65069786032G_n &= 89097229H_{n+1} - 59590077H_n + 66767796H_{n-1} - 396\,357741H_{n-2} \\ &\quad -1113744728H_{n-3} + 4861864449H_{n-4}. \end{aligned}$$

Proof. Note that all the identities hold for all integers $n$. We prove (5.1). To show (5.1), writing

$$H_n = a \times G_{n+6} + b \times G_{n+5} + c \times G_{n+4} + d \times G_{n+3} + e \times G_{n+2} + f \times G_{n+1}$$

and solving the system of equations

$$\begin{aligned}
H_0 &= a \times G_6 + b \times G_5 + c \times G_4 + d \times G_3 + e \times G_2 + f \times G_1 \\
H_1 &= a \times G_7 + b \times G_6 + c \times G_5 + d \times G_4 + e \times G_3 + f \times G_2 \\
H_2 &= a \times G_8 + b \times G_7 + c \times G_6 + d \times G_5 + e \times G_4 + f \times G_3 \\
H_3 &= a \times G_9 + b \times G_8 + c \times G_7 + d \times G_6 + e \times G_5 + f \times G_4 \\
H_4 &= a \times G_{10} + b \times G_9 + c \times G_8 + d \times G_7 + e \times G_6 + f \times G_5 \\
H_5 &= a \times G_{11} + b \times G_{10} + c \times G_9 + d \times G_8 + e \times G_7 + f \times G_6
\end{aligned}$$

we find that $a = -\frac{61}{169}, b = -\frac{21}{169}, c = \frac{1483}{169}, d = -\frac{956}{169}, e = -\frac{886}{169}, f = -\frac{863}{169}$. The other equalities can be proved similarly.

Secondly, we present a few basic relations between $\{G_n\}$ and $\{E_n\}$.

Lemma 9. *The following equalities are true:*

$$\begin{aligned}
169E_n &= 24G_{n+6} - 61G_{n+5} - 46G_{n+4} - 81G_{n+3} - 103G_{n+2} - 173G_{n+1}, \\
13E_n &= -G_{n+5} + 2G_{n+4} + 3G_{n+3} + 5G_{n+2} + 7G_{n+1} + 24G_n, \\
E_n &= G_n - G_{n-1},
\end{aligned}$$

*and*

$$\begin{aligned}
40G_n &= E_{n+6} - E_{n+5} - 4E_{n+4} - 9E_{n+3} - 16E_{n+2} - 27E_{n+1}, \\
40G_n &= E_{n+5} - E_{n+4} - 4E_{n+3} - 9E_{n+2} - 16E_{n+1} + 13E_n, \\
40G_n &= E_{n+4} - E_{n+3} - 4E_{n+2} - 9E_{n+1} + 24E_n + 13E_{n-1}, \\
40G_n &= E_{n+3} - E_{n+2} - 4E_{n+1} + 31E_n + 24E_{n-1} + 13E_{n-2}, \\
40G_n &= E_{n+2} - E_{n+1} + 36E_n + 31E_{n-1} + 24E_{n-2} + 13E_{n-3}, \\
40G_n &= E_{n+1} + 39E_n + 36E_{n-1} + 31E_{n-2} + 24E_{n-3} + 13E_{n-4}.
\end{aligned}$$

Note that all the identities in the above Lemma can be proved by induction as well.

Thirdly, we give a few basic relations between $\{H_n\}$ and $\{E_n\}$.

LEMMA 10. *The following equalities are true:*

$$
\begin{aligned}
65H_n &= -36E_{n+6} - 19E_{n+5} + 589E_{n+4} + 284E_{n+3} + 31E_{n+2} - 163E_{n+1}, \\
5H_n &= -7E_{n+5} + 37E_{n+4} + 8E_{n+3} - 17E_{n+2} - 43E_{n+1} - 36E_n, \\
5H_n &= 23E_{n+4} - 13E_{n+3} - 52E_{n+2} - 92E_{n+1} - 113E_n - 91E_{n-1}, \\
5H_n &= 33E_{n+3} + 17E_{n+2} + 23E_{n+1} + 48E_n + 162E_{n-1} + 299E_{n-2}, \\
5H_n &= 83E_{n+2} + 122E_{n+1} + 213E_n + 393E_{n-1} + 662E_{n-2} + 429E_{n-3}, \\
5H_n &= 288E_{n+1} + 462E_n + 808E_{n-1} + 1243E_{n-2} + 1342E_{n-3} + 1079E_{n-4},
\end{aligned}
$$

*and*

$$
\begin{aligned}
105738402302E_n &= 38647976H_{n+6} - 127766515H_{n+5} + 183710648H_{n+4} - 1268845658H_{n+3} \\
&\quad +2306044577H_{n+2} - 1561953023H_{n+1} \\
8133723254E_n &= -3882351H_{n+5} + 23050352H_{n+4} - 82738906H_{n+3} + 198198493H_{n+2} \\
&\quad -87448099H_{n+1} + 38647976H_n \\
8133723254E_n &= 15285650H_{n+4} - 94385959H_{n+3} + 178786738H_{n+2} - 114624556H_{n+1} \\
&\quad -4057885H_n - 50470563H_{n-1} \\
8133723254E_n &= -63814659H_{n+3} + 224643688H_{n+2} - 38196306H_{n+1} + 102941665H_n \\
&\quad +117671587H_{n-1} + 198713450H_{n-2} \\
8133723254E_n &= 97014370H_{n+2} - 229640283H_{n+1} - 216131630H_n - 329031026H_{n-1} \\
&\quad -503247799H_{n-2} - 829590567H_{n-3} \\
8133723254E_n &= -35611543H_{n+1} + 74911480H_n + 156040824H_{n-1} + 175852791H_{n-2} \\
&\quad +237567503H_{n-3} + 1261186810H_{n-4}
\end{aligned}
$$

We now present a few special identities for the modified 6-primes sequence $\{E_n\}$.

THEOREM 11. *(Catalan's identity) For all integers $n$ and $m$, the following identity holds*

$$
\begin{aligned}
E_{n+m}E_{n-m} - E_n^2 &= (G_{n+m} - G_{n+m-1})(G_{n-m} - G_{n-m-1}) - (G_n - G_{n-1})^2 \\
&= (G_n(G_m - G_{m+1}) + G_{n-1}(-G_m + G_{m-2}) + G_{n-2}(-G_m + G_{m-1})) \\
&\quad (G_n(G_{-m} - G_{1-m}) + G_{n-1}(-G_{-m} + G_{-m-2}) + G_{n-2}(-G_{-m} + G_{-m-1})) \\
&\quad -(G_n - G_{n-1})^2
\end{aligned}
$$

Proof. We use the identity

$$E_n = G_n - G_{n-1}.$$

Note that for $m = 1$ in Catalan's identity, we get the Cassini identity for the modified 6-primes sequnce

COROLLARY 12. *(Cassini's identity) For all integers numbers $n$ and $m$, the following identity holds*

$$E_{n+1}E_{n-1} - E_n^2 = (G_{n+1} - G_n)(G_{n-1} - G_{n-2}) - (G_n - G_{n-1})^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using $E_n = G_n - G_{n-1}$.The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified 6-primes sequence $\{E_n\}$.

THEOREM 13. *Let $n$ and $m$ be any integers. Then the following identities are true:*

**(a):** *(d'Ocagne's identity)*

$$E_{m+1}E_n - E_mE_{n+1} = (G_{m+1} - G_m)(G_n - G_{n-1}) - (G_m - G_{m-1})(G_{n+1} - G_n).$$

**(b):** *(Gelin-Cesàro's identity)*

$$E_{n+2}E_{n+1}E_{n-1}E_{n-2} - E_n^4 = (G_{n+2} - G_{n+1})(G_{n+1} - G_n)(G_{n-1} - G_{n-2})(G_{n-2} - G_{n-3}) - (G_n - G_{n-1})^4.$$

**(c):** *(Melham's identity)*

$$E_{n+1}E_{n+2}E_{n+6} - E_{n+3}^3 = (G_{n+1} - G_n)(G_{n+2} - G_{n+1})(G_{n+6} - G_{n+5}) - (G_{n+3} - G_{n+2})^3.$$

Proof. Use the identity $E_n = G_n - G_{n-1}$.

## 6. Linear Sums

The following Theorem presents some linear summing formulas of generalized Hexanacci numbers with positive subscripts.

THEOREM 14. *For $n \geq 0$ we have the following formulas:*

**(a):** *(Sum of the generalized Hexanacci numbers) If $r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - 1 \neq 0$ then*

$$\sum_{k=0}^{n} W_k = \frac{\Delta_1}{r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - 1}$$

*where*

$$\begin{aligned}
\Delta_1 \quad = \quad & W_{n+6} + (1 - r_1)W_{n+5} + (1 - r_1 - r_2)W_{n+4} + (1 - r_1 - r_2 - r_3)W_{n+3} \\
& +(1 - r_1 - r_2 - r_3 - r_4)W_{n+2} + (1 - r_1 - r_2 - r_3 - r_4 - r_5)W_{n+1} \\
& -W_5 + (r_1 - 1)W_4 + (r_1 + r_2 - 1)W_3 + (r_1 + r_2 + r_3 - 1)W_2 \\
& +(r_1 + r_2 + r_3 + r_4 - 1)W_1 + (r_1 + r_2 + r_3 + r_4 + r_5 - 1)W_0
\end{aligned}$$

**(b):** *If $(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - 1)(r_1 - r_2 + r_3 - r_4 + r_5 - r_6 + 1) \neq 0$ then*

$$\sum_{k=0}^{n} W_{2k} = \frac{\Delta_2}{(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - 1)(r_1 - r_2 + r_3 - r_4 + r_5 - r_6 + 1)}$$

*where*

$$
\begin{aligned}
\Delta_2 &= -(r_2+r_4+r_6-1)W_{2n+2}+(r_3+r_5+r_1(r_2+r_4+r_6))W_{2n+1} \\
&+(r_4+r_6+r_1(r_3+r_5)-r_2(r_4+r_6)+(r_3+r_5)^2-(r_4+r_6)^2)W_{2n} \\
&+(r_5-r_2r_5+(r_1+r_3)r_4+(r_1+r_3)r_6)W_{2n-1}+(r_6+(r_1+r_3)r_5-(r_2+r_4)r_6+r_5^2-r_6^2)W_{2n-2} \\
&+r_6(r_1+r_3+r_5)W_{2n-3}-(r_1+r_3+r_5)W_5+(r_2+r_4+r_6+(r_1+r_3+r_5)r_1-1)W_4 \\
&+((r_3+r_5)r_2-(r_4+r_6)r_1-r_3-r_5)W_3 \\
&+(r_4+r_6+(r_1+r_3)r_5-(r_4+r_6)r_2+(r_1+r_3)^2-(r_2-1)^2)W_2+(-r_5+(r_2+r_4)r_5-(r_1+r_3)r_6)W_1 \\
&+(2r_2+2r_4+2r_1r_5+2r_3r_5-r_2r_6+r_6-r_4r_6+(r_1+r_3)^2-(r_2+r_4)^2+r_5^2-1)W_0
\end{aligned}
$$

**(c):**

$$\sum_{k=0}^{n} W_{2k+1} = \frac{\Delta_3}{(r_1+r_2+r_3+r_4+r_5+r_6-1)(r_1-r_2+r_3-r_4+r_5-r_6+1)}$$

*where*

$$
\begin{aligned}
\Delta_3 &= (r_1+r_3+r_5)W_{2n+2}+((r_2+r_4+r_6)-(r_2+r_4+r_6)^2+(r_3+r_5)^2+r_1(r_3+r_5))W_{2n+1} \\
&+((1-r_2)(r_3+r_5)+r_1(r_4+r_6))W_{2n}+((r_1+r_3)r_5+r_5^2-(r_4+r_6)r_2+(r_4+r_6)-(r_4+r_6)^2)W_{2n-1} \\
&+((1-(r_2+r_4))r_5+(r_1+r_3)r_6)W_{2n-2}-r_6(r_2+r_4+r_6-1)W_{2n-3}+(r_2+r_4+r_6-1)W_5 \\
&-((r_3+r_5)+(r_2+r_4+r_6)r_1)W_4+(2r_2+r_4+r_6+r_1r_3+r_1r_5-r_2r_4-r_2r_6+r_1^2-r_2^2-1)W_3 \\
&-((1-r_2)r_5+(r_1+r_3)(r_4+r_6))W_2 \\
&+(2r_2+2r_4+r_6+r_1r_5+r_3r_5-r_2r_6-r_4r_6+r_1^2+r_3^2+2r_1r_3-r_2^2-r_4^2-2r_2r_4-1)W_1-r_6(r_1+r_3+r_5)W_0
\end{aligned}
$$

Proof. The proof is given in Soykan [7].

The following proposition presents some formulas of generalized 6-primes numbers with positive subscripts.

PROPOSITION 15. *If $r_1=2, r_2=3, r_3=5, r_4=7, r_5=11, r_6=13$ then for $n \geq 0$ we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} V_k = \frac{1}{40}(V_{n+6}-V_{n+5}-4V_{n+4}-9V_{n+3}-16V_{n+2}-27V_{n+1}-V_5+V_4+4V_3+9V_2+16V_1+27V_0)$.
- **(b):** $\sum_{k=0}^{n} V_{2k} = \frac{1}{80}(11V_{2n+2}-31V_{2n+1}+76V_{2n}-59V_{2n-1}+44V_{2n-2}-117V_{2n-3}+9V_5-29V_4+4V_3-41V_2-4V_1-63V_0)$.
- **(c):** $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{80}(-9V_{2n+2}+109V_{2n+1}-4V_{2n}+121V_{2n-1}+4V_{2n-2}+143V_{2n-3}-11V_5+31V_4+4V_3+59V_2+36V_1+117V_0)$.

Proof. Take $r_1=2, r_2=3, r_3=5, r_4=7, r_5=11$ in Theorem 14.

As special cases of above proposition, we have the following three corollaries. First one presents some summing formulas of 6-primes numbers (take $V_n=G_n$ with $G_0=0, G_1=0, G_2=0, G_3=0, G_4=1, G_5=2$).

COROLLARY 16. *For $n \geq 0$ we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} G_k = \frac{1}{40}(G_{n+6}-G_{n+5}-4G_{n+4}-9G_{n+3}-16G_{n+2}-27G_{n+1}-1)$.
- **(b):** $\sum_{k=0}^{n} G_{2k} = \frac{1}{80}(11G_{2n+2}-31G_{2n+1}+76G_{2n}-59G_{2n-1}+44G_{2n-2}-117G_{2n-3}-11)$.
- **(c):** $\sum_{k=0}^{n} G_{2k+1} = \frac{1}{80}(-9G_{2n+2}+109G_{2n+1}-4G_{2n}+121G_{2n-1}+4G_{2n-2}+143G_{2n-3}+9)$.

Second one presents some summing formulas of Lucas 6-primes numbers (take $G_n = H_n$ with $H_0 = 6,\ H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542$).

Corollary 17. *For $n \geq 0$ we have the following formulas:*

**(a):** $\sum_{k=0}^{n} H_k = \frac{1}{40}(H_{n+6} - H_{n+5} - 4H_{n+4} - 9H_{n+3} - 16H_{n+2} - 27H_{n+1} + 56)$.

**(b):** $\sum_{k=0}^{n} H_{2k} = \frac{1}{80}(11H_{2n+2} - 31H_{2n+1} + 76H_{2n} - 59H_{2n-1} + 44H_{2n-2} - 117H_{2n-3} - 104)$

**(c):** $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{80}(-9H_{2n+2} + 109H_{2n+1} - 4H_{2n} + 121H_{2n-1} + 4H_{2n-2} + 143H_{2n-3} + 216)$.

Third one presents some summing formulas of modified 6-primes numbers (take $H_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1$).

Corollary 18. *For $n \geq 0$ we have the following formulas:*

**(a):** $\sum_{k=0}^{n} E_k = \frac{1}{40}(E_{n+6} - E_{n+5} - 4E_{n+4} - 9E_{n+3} - 16E_{n+2} - 27E_{n+1})$.

**(b):** $\sum_{k=0}^{n} E_{2k} = \frac{1}{80}(11E_{2n+2} - 31E_{2n+1} + 76E_{2n} - 59E_{2n-1} + 44E_{2n-2} - 117E_{2n-3} - 20)$.

**(c):** $\sum_{k=0}^{n} E_{2k+1} = \frac{1}{80}(-9E_{2n+2} + 109E_{2n+1} - 4E_{2n} + 121E_{2n-1} + 4E_{2n-2} + 143E_{2n-3} + 20)$.

The following Theorem presents some linear summing formulas of generalized Hexanacci numbers with negative subscripts.

Theorem 19. *For $n \geq 1$ we have the following formulas:*

**(a):** *(Sum of the generalized Hexanacci numbers with negative indices) If* $r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - 1 \neq 0$, *then*

$$\sum_{k=1}^{n} W_{-k} = \frac{\Delta_4}{r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - 1}$$

*where*

$$\begin{aligned}
\Delta_4 &= -W_{-n+5} + (r_1 - 1)W_{-n+4} + (r_1 + r_2 - 1)W_{-n+3} + (r_1 + r_2 + r_3 - 1)W_{-n+2} \\
&\quad +(r_1 + r_2 + r_3 + r_4 - 1)W_{-n+1} + (r_1 + r_2 + r_3 + r_4 + r_5 - 1)W_{-n} \\
&\quad +W_5 + (1 - r_1)W_4 + (1 - r_1 - r_2)W_3 + (1 - r_1 - r_2 - r_3)W_2 \\
&\quad +(1 - r_1 - r_2 - r_3 - r_4)W_1 + (1 - r_1 - r_2 - r_3 - r_4 - r_5)W_0
\end{aligned}$$

**(b):** *If* $(r_1 - r_2 + r_3 - r_4 + r_5 - r_6 + 1)(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - 1) \neq 0$ *then*

$$\sum_{k=1}^{n} W_{-2k} = \frac{\Delta_5}{(r_1 - r_2 + r_3 - r_4 + r_5 - r_6 + 1)(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - 1)}$$

*where*

$$\begin{aligned}\Delta_5 &= (r_2+r_4+r_6-1)W_{-2n+4}-(r_3+r_5+(r_2+r_4+r_6)r_1)W_{-2n+3}\\ &+(r_4+r_6-(r_4+r_6)r_2+(r_3+r_5+r_1)r_1-(r_2-1)^2)W_{-2n+2}\\ &-((r_1+r_3)r_4+(r_1+r_3)r_6+(1-r_2)r_5)W_{-2n+1}\\ &+(r_6+2(r_2+r_4)+(r_1+r_3)r_5-(r_2+r_4)r_6+(r_1+r_3)^2-(r_2+r_4)^2-1)W_{-2n}\\ &-r_6(r_1+r_3+r_5)W_{-2n-1}+(r_1+r_3+r_5)W_5-(r_2+r_4+r_6+(r_3+r_5+r_1)r_1-1)W_4\\ &+((r_4+r_6)r_1+(1-r_2)(r_3+r_5))W_3+(-r_4-r_6-(r_1+r_3)r_5+(r_4+r_6)r_2-(r_1+r_3)^2+(r_2-1)^2)W_2\\ &+(r_5+(r_1+r_3)r_6-(r_2+r_4)r_5)W_1\\ &+(-r_6-2(r_2+r_4)-2(r_1+r_3)r_5+(r_2+r_4)r_6-(r_1+r_3)^2+(r_2+r_4)^2-r_5^2+1)W_0\end{aligned}$$

**(c):**

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{\Delta_6}{(r_1-r_2+r_3-r_4+r_5-r_6+1)(r_1+r_2+r_3+r_4+r_5+r_6-1)}$$

*where*

$$\begin{aligned}\Delta_6 &= -(r_1+r_3+r_5)W_{-2n+4}+(r_2+r_4+r_6+(r_3+r_5+r_1)r_1-1)W_{-2n+3}\\ &+((r_2-1)(r_3+r_5)-(r_4+r_6)r_1)W_{-2n+2}\\ &+(r_4+r_6+(r_1+r_3)r_5-(r_4+r_6)r_2+(r_1+r_3)^2-(r_2-1)^2)W_{-2n+1}\\ &+(-r_5-(r_1+r_3)r_6+(r_2+r_4)r_5)W_{-2n}+r_6(r_2+r_4+r_6-1)W_{-2n-1}-(r_2+r_4+r_6-1)W_5\\ &+(r_3+r_5+(r_2+r_4+r_6)r_1)W_4+(-r_4-r_6-(r_1+r_3+r_5)r_1+(r_4+r_6)r_2+(r_2-1)^2)W_3\\ &+((r_4+r_6)r_1+(r_4+r_6)r_3+(1-r_2)r_5)W_2\\ &+(-r_6-2(r_2+r_4)-(r_1+r_3)r_5+(r_2+r_4)r_6-(r_1+r_3)^2+(r_2+r_4)^2+1)W_1+r_6(r_1+r_3+r_5)W_0\end{aligned}$$

Proof. The proof is given in Soykan [7].

The following proposition presents some formulas of generalized 6-primes numbers with negative subscripts.

PROPOSITION 20. *If* $r_1=2, r_2=3, r_3=5, r_4=7, r_5=11, r_6=13$ *then for* $n\geq 1$ *we have the following formulas:*

**(a):** $\sum_{k=1}^{n} V_{-k} = \frac{1}{40}(-V_{-n+5}+V_{-n+4}+4V_{-n+3}+9V_{-n+2}+16V_{-n+1}+27V_{-n}+V_5-V_4-4V_3-9V_2-16V_1-27V_0)$.

**(b):** $\sum_{k=1}^{n} G_{-2k} = \frac{1}{80}(-11V_{-2n+4}+31V_{-2n+3}+4V_{-2n+2}+59V_{-2n+1}+36V_{-2n}+117V_{-2n-1}-9V_5+29V_4-4V_3+41V_2+4V_1+63V_0)$.

**(c):** $\sum_{k=1}^{n} G_{-2k+1} = \frac{1}{80}(9V_{-2n+4}-29V_{-2n+3}+4V_{-2n+2}-41V_{-2n+1}-4V_{-2n}-143V_{-2n-1}+11V_5-31V_4-4V_3-59V_2-36V_1-117V_0)$.

Proof. Take $r_1=2, r_2=3, r_3=5, r_4=7, r_5=11$ in Theorem 19.

From the above proposition, we have the following corollary which gives sum formulas of 6-primes numbers (take $G_n=G_n$ with $G_0=0, G_1=0, G_2=0, G_3=0, G_4=1, G_5=2$).

COROLLARY 21. *For* $n\geq 1$*, 6-primes numbers have the following properties.*

**(a):** $\sum_{k=1}^{n} G_{-k} = \frac{1}{40}(-G_{-n+5}+G_{-n+4}+4G_{-n+3}+9G_{-n+2}+16G_{-n+1}+27G_{-n}+1)$.

**(b):** $\sum_{k=1}^{n} G_{-2k} = \frac{1}{80}(-11G_{-2n+4} + 31G_{-2n+3} + 4G_{-2n+2} + 59G_{-2n+1} + 36G_{-2n} + 117G_{-2n-1} + 11).$

**(c):** $\sum_{k=1}^{n} G_{-2k+1} = \frac{1}{80}(9G_{-2n+4} - 29G_{-2n+3} + 4G_{-2n+2} - 41G_{-2n+1} - 4G_{-2n} - 143G_{-2n-1} - 9).$

Taking $G_n = H_n$ with $H_0 = 6,\ H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542$ in the last proposition, we have the following corollary which presents sum formulas of 6-primes -Lucas numbers.

Corollary 22. *For $n \geq 1$, 6-primes -Lucas numbers have the following properties.*

**(a):** $\sum_{k=1}^{n} H_{-k} = \frac{1}{40}(-H_{-n+5} + H_{-n+4} + 4H_{-n+3} + 9H_{-n+2} + 16H_{-n+1} + 27H_{-n} - 56).$

**(b):** $\sum_{k=1}^{n} H_{-2k} = \frac{1}{80}(-11H_{-2n+4} + 31H_{-2n+3} + 4H_{-2n+2} + 59H_{-2n+1} + 36H_{-2n} + 117H_{-2n-1} + 104).$

**(c):** $\sum_{k=1}^{n} H_{-2k+1} = \frac{1}{80}(9H_{-2n+4} - 29H_{-2n+3} + 4H_{-2n+2} - 41H_{-2n+1} - 4H_{-2n} - 143H_{-2n-1} - 216).$

From the above proposition, we have the following corollary which gives sum formulas of modified 6-primes numbers (take $H_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1$).

Corollary 23. *For $n \geq 1$, modified 6-primes numbers have the following properties.*

**(a):** $\sum_{k=1}^{n} E_{-k} = \frac{1}{40}(-E_{-n+5} + E_{-n+4} + 4E_{-n+3} + 9E_{-n+2} + 16E_{-n+1} + 27E_{-n}).$

**(b):** $\sum_{k=1}^{n} E_{-2k} = \frac{1}{80}(-11E_{-2n+4} + 31E_{-2n+3} + 4E_{-2n+2} + 59E_{-2n+1} + 36E_{-2n} + 117E_{-2n-1} + 20).$

**(c):** $\sum_{k=1}^{n} E_{-2k+1} = \frac{1}{80}(9E_{-2n+4} - 29E_{-2n+3} + 4E_{-2n+2} - 41E_{-2n+1} - 4E_{-2n} - 143E_{-2n-1} - 20).$

## 7. Matrices Related with Generalized 6-primes Numbers

Matrix formulation of $W_n$ can be given as

$$\begin{pmatrix} W_{n+5} \\ W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_5 \\ W_4 \\ W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix} \tag{7.1}$$

For matrix formulation (7.1), see [2]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \\ W_{n+4} \\ W_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \end{pmatrix}.$$

We define the square matrix $A$ of order 6 as:

$$A=\begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A=-13$. From (1.4) we have

$$\begin{pmatrix} V_{n+5} \\ V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix}=\begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix}. \tag{7.2}$$

and from (7.1) (or using (7.2) and induction) we have

$$\begin{pmatrix} V_{n+5} \\ V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix}=\begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n\begin{pmatrix} V_5 \\ V_4 \\ V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V_n=G_n$ in (7.2) we have

$$\begin{pmatrix} G_{n+5} \\ G_{n+4} \\ G_{n+3} \\ G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix}=\begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} G_{n+4} \\ G_{n+3} \\ G_{n+2} \\ G_{n+1} \\ G_n \\ G_{n-1} \end{pmatrix}. \tag{7.3}$$

We also define

$$B_n=\begin{pmatrix}
G_{n+4} & \sum\limits_{k=0}^{4} r_{k+2}G_{n+3-k} & \sum\limits_{k=0}^{3} r_{k+3}G_{n+3-k} & \sum\limits_{k=0}^{2} r_{k+4}G_{n+3-k} & \sum\limits_{k=0}^{1} r_{k+5}G_{n+3-k} & r_6G_{n+3} \\
G_{n+3} & \sum\limits_{k=1}^{5} r_{k+1}G_{n+3-k} & \sum\limits_{k=1}^{4} r_{k+2}G_{n+3-k} & \sum\limits_{k=1}^{3} r_{k+3}G_{n+3-k} & \sum\limits_{k=1}^{2} r_{k+4}G_{n+3-k} & r_6G_{n+2} \\
G_{n+2} & \sum\limits_{k=2}^{6} r_{k}G_{n+3-k} & \sum\limits_{k=2}^{5} r_{k+1}G_{n+3-k} & \sum\limits_{k=2}^{4} r_{k+2}G_{n+3-k} & \sum\limits_{k=2}^{3} r_{k+3}G_{n+3-k} & r_6G_{n+1} \\
G_{n+1} & \sum\limits_{k=3}^{7} r_{k-1}G_{n+3-k} & \sum\limits_{k=3}^{6} r_{k}G_{n+3-k} & \sum\limits_{k=3}^{5} r_{k+1}G_{n+3-k} & \sum\limits_{k=3}^{4} r_{k+2}G_{n+3-k} & r_6G_{n} \\
G_{n} & \sum\limits_{k=4}^{8} r_{k-2}G_{n+3-k} & \sum\limits_{k=4}^{7} r_{k-1}G_{n+3-k} & \sum\limits_{k=4}^{6} r_{k}G_{n+3-k} & \sum\limits_{k=4}^{5} r_{k+1}G_{n+3-k} & r_6G_{n-1} \\
G_{n-1} & \sum\limits_{k=5}^{9} r_{k-3}G_{n+3-k} & \sum\limits_{k=5}^{8} r_{k-2}G_{n+3-k} & \sum\limits_{k=5}^{7} r_{k-1}G_{n+3-k} & \sum\limits_{k=5}^{6} r_{k}G_{n+3-k} & r_6G_{n-2}
\end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+4} & \sum_{k=0}^{4} r_{k+2}V_{n+3-k} & \sum_{k=0}^{3} r_{k+3}V_{n+3-k} & \sum_{k=0}^{2} r_{k+4}V_{n+3-k} & \sum_{k=0}^{1} r_{k+5}V_{n+3-k} & r_6V_{n+3} \\ V_{n+3} & \sum_{k=1}^{5} r_{k+1}V_{n+3-k} & \sum_{k=1}^{4} r_{k+2}V_{n+3-k} & \sum_{k=1}^{3} r_{k+3}V_{n+3-k} & \sum_{k=1}^{2} r_{k+4}V_{n+3-k} & r_6V_{n+2} \\ V_{n+2} & \sum_{k=2}^{6} r_{k}V_{n+3-k} & \sum_{k=2}^{5} r_{k+1}V_{n+3-k} & \sum_{k=2}^{4} r_{k+2}V_{n+3-k} & \sum_{k=2}^{3} r_{k+3}V_{n+3-k} & r_6V_{n+1} \\ V_{n+1} & \sum_{k=3}^{7} r_{k-1}V_{n+3-k} & \sum_{k=3}^{6} r_{k}V_{n+3-k} & \sum_{k=3}^{5} r_{k+1}V_{n+3-k} & \sum_{k=3}^{4} r_{k+2}V_{n+3-k} & r_6V_{n} \\ V_{n} & \sum_{k=4}^{8} r_{k-2}V_{n+3-k} & \sum_{k=4}^{7} r_{k-1}V_{n+3-k} & \sum_{k=4}^{6} r_{k}V_{n+3-k} & \sum_{k=4}^{5} r_{k+1}V_{n+3-k} & r_6V_{n-1} \\ V_{n-1} & \sum_{k=5}^{9} r_{k-3}V_{n+3-k} & \sum_{k=5}^{8} r_{k-2}V_{n+3-k} & \sum_{k=5}^{7} r_{k-1}V_{n+3-k} & \sum_{k=5}^{6} r_{k}V_{n+3-k} & r_6V_{n-2} \end{pmatrix}.$$

where

$$r_1 = 2, r_2 = 3, r_3 = 5, r_4 = 7, r_5 = 11, r_6 = 13.$$

THEOREM 24. *For all integer $m, n \geq 0$, we have*

**(a):** $B_n = A^n$

**(b):** $C_1 A^n = A^n C_1$

**(c):** $C_{n+m} = C_n B_m = B_m C_n$.

**Proof.**

**(a):** By expanding the vectors on the both sides of (7.3) to 6-colums and multiplying the obtained on the right-hand side by $A$, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

**(b):** Using (a) and definition of $C_1$, (b) follows.

**(c):** We have $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^m C_1 = A^{n-1}C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

Some properties of matrix $A^n$ can be given as

$$A^n = 2A^{n-1} + 3A^{n-2} + 5A^{n-3} + 7A^{n-4} + 11A^{n-5} + 13A^{n-6}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = (-13)^n$$

for all integer $m$ and $n$.

THEOREM 25. *For $m, n \geq 0$ we have*

$$\begin{aligned} V_{n+m} &= V_n G_{m+4} + \sum_{i=1}^{6-1} V_{n-i} \left( \sum_{j=1}^{6-i} r_{j+i} G_{m+4-j} \right) \qquad (7.4) \\ &= V_n G_{m+4} + V_{n-1}(3G_{m+3} + 5G_{m+2} + 7G_{m+1} + 11G_m + 13G_{m-1}) \\ &\quad + V_{n-2}(5G_{m+3} + 7G_{m+2} + 11G_{m+1} + 13G_m) + V_{n-3}(7G_{m+3} + 11G_{m+2} + 13G_{m+1}) \\ &\quad + V_{n-4}(11G_{m+3} + 13G_{m+2}) + 13V_{n-5}G_{m+3}. \end{aligned}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of $C_{n+m}$ is the product of row $C_n$ and a column $B_m$. From the last equation we say that an element of $C_{n+m}$ is the product of a row $C_n$ and column $B_m$. We just compare the linear combination of the 2nd row and 1st column entries of the matrices $C_{n+m}$ and $C_n B_m$. This completes the proof.

REMARK 26. *By induction, it can be proved that for all integers $m, n \leq 0$, (7.4) holds. So for all integers $m, n$, (7.4) is true.*

COROLLARY 27. *For all integers $m, n$, we have*

$$G_{n+m} = G_n G_{m+4} + \sum_{i=1}^{6-1} G_{n-i} \left( \sum_{j=1}^{6-i} r_{j+i} G_{m+4-j} \right), \qquad (7.5)$$

$$H_{n+m} = H_n G_{m+4} + \sum_{i=1}^{6-1} H_{n-i} \left( \sum_{j=1}^{6-i} r_{j+i} G_{m+4-j} \right), \qquad (7.6)$$

$$E_{n+m} = E_n G_{m+4} + \sum_{i=1}^{6-1} E_{n-i} \left( \sum_{j=1}^{6-i} r_{j+i} G_{m+4-j} \right). \qquad (7.7)$$

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