Relative Extension of Continuous Mappings

by

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ABSTRACT. In this paper, the notion of a relative extension of continuous mappings is defined. The relative extension of continuous mappings is the generalization of the notion of a relative retract in topological spaces. The relative extension of continuous mappings will be applied to fixed point theory.

1 Introduction

In [9] we have introduced the notion of a relative retract in metric spaces and defined the class of absolute relative retracts (ARR) and absolute neighborhood retract ($ANRR$). The relative retracts are an essential generalization of the retracts in the sense of Borsuk. In papers [9, 10, 11] their properties are studied with the use of new topological tools (relative homotopy, relative contractability). Relative retracts applied to fixed point theory, to the theory of coincidence (see [9, 10]) and to the study of global and local properties of metrizable spaces (see [11]). In this paper we define the relative extension of continuous maps, that is, the generalization of the notion of relative retracts in topological spaces (not necessarily metrizable). We also define two classes of topological spaces: relative extension (ESR) and relative neighborhood extension ($NESR$). These spaces are an essential generalization of spaces considered by G. Fournier and A. Granas in [4]. The relative extension of continuous maps is applied to the theorems on the fixed points of multivalued compact and noncompact maps. This article is an integral part of relative retracts theory (see [9, 10, 11]).

2 Preliminaries

Throughout this paper, all spaces are assumed to be Hausdorff topological spaces and all singlevalued mappings are continuous. A continuous map $f : X \to Y$ is called perfect, if for every $y \in Y$, the set $f^{-1}(y)$ is nonempty and compact provided f is closed. Let X and Y be two spaces and assume that for every $x \in X$ a nonempty subset $\varphi(x)$ of Y is given. In this case, we say that $\varphi : X \multimap Y$ is a multivalued mapping. Let H_* be the *Čech* homology functor with compact carriers, coefficients in the field of rational numbers $\mathbb{Q}$ from the category of Hausdorff topological spaces, continuous maps to the category of a graded vector space and linear maps of degree zero. Thus $H_*(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being a q-dimensional *Čech* homology group with compact carriers of X. For a continuous map $f : X \to Y$, $H_*(f)$ is the induced linear map $f_* = \{f_q\}$, where $f_q : H_q(X) \to H_q(Y)$ ([5]). A set X is acyclic if:

(i) X is nonempty,

(ii) $H_q(X) = 0$ for every $q \geq 1$ and

(iii) $H_0(X) \approx \mathbb{Q}$.

Let $u : E \to E$ be an endomorphism of an arbitrary vector space. Suppose

$$N(u) = \{x \in E : u^n(x) = 0 \text{ for some } n\},$$

where u^n is the n-th iterate of u and $\widetilde{E} = E/N(u)$. Since $u(N(u)) \subset N(u)$, we have the induced endomorphism $\widetilde{u} : \widetilde{E} \to \widetilde{E}$ defined by $\widetilde{u}([x]) = [u(x)]$. We call u admissible provided $dim\widetilde{E} < \infty$. Let $u = \{u_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call u a Leray endomorphism if

(i) all u_q are admissible,

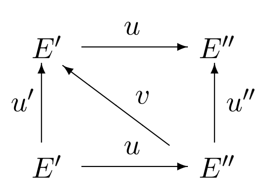
(ii) almost all $\widetilde{E_q}$ are trivial.

For such a u, we define the (generalized) Lefschetz number $\Lambda(u)$ of u by putting

$$\Lambda(u) = \sum_q (-1)^q tr(\widetilde{u_q}),$$

where $tr(\widetilde{u_q})$ is the ordinary trace of $\widetilde{u_q}$ (comp. [5]). The following important property of a Leray endomorphism is a consequence of a well-known formula $tr(u \circ v) = tr(v \circ u)$ for the ordinary trace.

Proposition 2.1. *([5])* Assume that, in the category of graded vector spaces the following diagram commutes



If one of u', u'' is a Leray endomorphism, then so is the other; and $\Lambda(u') = \Lambda(u'')$.

An endomorphism $u : E \to E$ of a graded vector space E is called weakly nilpotent if for every $q \geq 0$ and for every $x \in E_q$, there exists an integer n such that $u_q^n(x) = 0$. For a weakly nilpotent endomorphism $u : E \to E$, we have $N(u) = E$.

Proposition 2.2. If $u : E \to E$ is a weakly nilpotent endomorphism, then $\Lambda(u) = 0$.

A perfect map $p : X \to Y$ is called Vietoris provided for every $y \in Y$ the set $p^{-1}(y)$ is acyclic. We recall that the composition of two Vietoris mappings is a Vietoris mapping and if $p : X \to Y$ is a Vietoris map then $p_* : H_*(X) \to H_*(Y)$ is an isomorphism (see, [5]). Let $\varphi : X \multimap Y$ be a multivalued map. We recall that the map φ is admissible (s-admissible) (see, [5]) if there exist a Vietoris map $p : Z \to X$ and a continuous map $q : Z \to Y$ such that for each $x \in X$

$$q(p^{-1}(x)) \subset \varphi(x) \ \ (q(p^{-1}(x)) = \varphi(x)) \ \ \text{(we will write } (p, q) \subset \varphi \ \ ((p, q) = \varphi)).$$

Let $\varphi : X \multimap Y$ be a map and let $A \subset X$ be a nonempty set. We denote $\varphi_A : A \multimap X$ a map given by the formula $\varphi_A(x) = \varphi(x)$ for each $x \in A$.

Definition 2.3. A topological vector space is called Klee admissible provided for every compact $K \subset E$ and for every open neighborhood of zero V in E there exists a continuous map $\pi_V : K \to E$ such that:
(2.3.1) $(x - \pi_V(x)) \in V$ *for every* $x \in K$ *and*
(2.3.2) there exists a natural number $n = n_K$ such that $\pi_V(K) \subset E^n$, where E^n is an n-dimensional subspace of E.

It is well known that any locally convex space is Klee admissible. We recall that a multivalued map $\varphi : X \multimap Y$ is compact, if the set $\overline{\varphi(X)} \subset Y$ is compact.

Theorem 2.4. *([5])* Let E be a Klee admissible space and let $U \subset E$ be an open set. Consider a diagram:

$$U \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } U,$$

p is Vietoris and q is compact. Then $q_* \circ p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_* \circ p_*^{-1}) \neq 0$ implies that p and q have a coincidence point, that is, there is a point $z \in Z$ such that $p(z) = q(z)$.

Theorem 2.5. *([6])* Let X be normal, $A \subset X$ closed, and $F_0 : A \to E$ a compact map into a normed space E. Then F_0 is extendable to a compact map $F : X \to E$.

Proposition 2.6. Let X be normal, $A \subset X$ closed, and $F_0 : A \to U$ a compact map into an open set $U \subset E$, where E is a normed space. Then F_0 is extendable to a compact map $F : V \to U$, where $V \subset X$ is some open neighborhood of A.

Proof. Let $F_0 : A \to U$ be a compact map, that is, $\overline{F_0(A)} \subset U$ is a compact set, where $A \subset X$ is closed and $U \subset E$ is an open set. There exists an open neighborhood $V_1 \subset U$ of $\overline{F_0(A)}$ such that $\overline{V}_1 \subset U$. From Theorem 2.5 there exists a compact extension $F : X \to E$ of $F_0 : A \to U \subset E$. Let $V = F^{-1}(V_1)$. We define a map $\widetilde{F} : V \to U$ by the formula $\widetilde{F}(x) = F(x)$ for each $x \in V$. We observe that the map $\widetilde{F}$ is a compact extension of F_0 and the proof is complete. □

Definition 2.7. *(see, [4])* A space $\mathbb{M}$ is an extension (resp. neighborhood extension) space provided for any compact pair (X, A) with $A \subset X$ closed and any map $f_0 : A \to \mathbb{M}$ there is an extension $f : X \to \mathbb{M}$ (resp. neighborhood extension $f : U \to \mathbb{M}$) of f_0 over X (resp. over the neighborhood of A in X). The classes of the extension spaces and the neighborhood extension spaces will be denoted by ES (written $\mathbb{M} \in ES$) and NES (written $\mathbb{M} \in NES$) respectively.

Let $X \in ANR$ and let $Y \subset X$ be a compact and nonempty subset. We recall that Y is movable in X provided every neighborhood U of Y admits a neighborhood V of Y, $V \subset U$, such that for every neighborhood W of Y, $W \subset V$, there exists a homotopy $H \colon V \times [0, 1] \to U$ with

$$H(x, 0) = x \quad \text{and} \quad H(x, 1) \in W, \quad \text{for any} \quad x \in V. \tag{1}$$

Let $y_0 \in Y$. We recall that Y has a trivial shape in X provided every neighborhood U of Y admits a neighborhood V of Y, $V \subset U$, such that there exists a homotopy $H \colon V \times [0, 1] \to U$ with

$$H(x, 0) = x \quad \text{and} \quad H(x, 1) = y_0, \quad \text{for any} \quad x \in V. \tag{2}$$

Let Y be a compact and metrizable space. We say that Y is movable provided there exists a space $X \in ANR$ and an embedding $h : Y \to X$ such that $h(Y)$ is movable in X.

Remark 2.8. We recall that in the metrizable spaces the property of movable is an absolute property, that is, if a compact set Y is movable in some ANR X and $h : Y \to X'$ is an embedding into an ANR X', then $h(Y)$ is movable in X' (see, [1]).

Remark 2.9. *([1])* The following are types of movable spaces: AR, ANR, FAR and $FANR$.

3 The families of sets

In this section, we will give the necessary notions through definitions and lemmas.

Definition 3.1. (**Trivial shape**) Let $\mathbb{T}$ be a Tychonoff cube, $X \subset \mathbb{T}$ be a compact space and let $x_0 \in X$ be an arbitrary point. We will say that X has a trivial shape in $\mathbb{T}$ provided every neighborhood $U \subset \mathbb{T}$ of X admits a neighborhood V of X, $V \subset U$, such that there exists a homotopy $H \colon V \times [0, 1] \to U$ with

$$H(x, 0) = x \quad \textit{and} \quad H(x, 1) = x_0, \quad \text{for any} \quad x \in V, \quad (\text{see } (2)).$$

A compact space X has a trivial shape if there exists a Tychonoff cube $\mathbb{T}$ and an embedding $h : X \to \mathbb{T}$ such that $h(X)$ has a trivial shape in $\mathbb{T}$.

Lemma 3.2. Let S be a nonempty set and let $X = \prod_{s \in S} X_s$. Assume that for each $s \in S$ there exists a Tychonoff cube $\mathbb{T}_s$ such that $X_s \subset \mathbb{T}_s$. The space X has a trivial shape in $\mathbb{T} = \prod_{s \in S} \mathbb{T}_s$ if and only if for each $s \in S$ a space X_s has a trivial shape in $\mathbb{T}_s$.

Proof. Assume that X has a trivial shape in $\mathbb{T}$. Let $\lambda \in X$, $\lambda = \{\lambda_s\}$ and let $a \in S$. We show that X_a has a trivial shape in $\mathbb{T}_a$. Let $U_a \subset \mathbb{T}_a$ be a compact neighborhood of X_a. The set $U = \prod_{s\in S} U_s$, where $U_s = \mathbb{T}_s$ for each $s \neq a$ is a compact neighborhood of X in $\mathbb{T} = \prod_{s\in S} \mathbb{T}_s$. There exists a compact neighborhood $V = \prod_{s\in S} V_s \subset U$ of X, $V_s = \mathbb{T}_s$ for each $s \notin \{s_1, s_2, .., s_n, a\}$ such that there exists a homotopy $H_\lambda \colon V \times [0,1] \to U$ with

$$H_\lambda(x,0) = x \text{ and } H_\lambda(x,1) = \lambda, \text{ for any } x \in V.$$

Suppose that

$$V_a \times [0,1] \xrightarrow{h} V \times [0,1] \xrightarrow{H_\lambda} U \xrightarrow{\pi} U_a,$$

where h is a homeomorphism given by the formula $h(v_a, t) = (\{y_s\}, t)$, $y_a = v_a$, $y_s = \lambda_s$ for $s \neq a$ and π is a projection, then we define a homotopy $H_a : V_a \times [0,1] \to U_a$ by the formula:

$$H_a = \pi \circ H_\lambda \circ h.$$

Assume now that for each $s \in S$ a space X_s has a trivial shape in $\mathbb{T}_s$. Let $\lambda \in X$, $\lambda = \{\lambda_s\}$ be an arbitrary point and let $U = \prod_{s\in S} U_s \subset \mathbb{T}$ be a compact neighborhood of X in $\mathbb{T}$ such that for each $s \notin \{s_1, s_2, ..., s_n\}$, $U_s = \mathbb{T}_s$. It follows that, for each $k = 1, ..., n$ there exists a compact neighborhood $V_{s_k} \subset U_{s_k}$ of X_{s_k} and a homotopy $H_{s_k} \colon V_{s_k} \times [0,1] \to U_{s_k}$ with

$$H_{s_k}(x,0) = x \text{ and } H_{s_k}(x,1) = \lambda_{s_k}, \text{ for any } x \in V_{s_k}.$$

Let $V = \prod_{s\in S} V_s$, $V_s = \mathbb{T}_s$ for any $s \notin \{s_1, s_2, ..., s_n\}$. We define a homotopy $H_\lambda : V \times [0,1] \to U$ by the formula

$$H_\lambda(\{v_s\}, t) = \{H_s(v_s, t)\}, \text{ for any } v = \{v_s\} \in V,$$

where

$$H_s : V_s \times [0,1] = \mathbb{T}_s \times [0,1] \to \mathbb{T}_s = U_s$$

for each $s \notin \{s_1, s_2, ..., s_n\}$ is a homotopy such that

$$H_s(v_s, 0) = v_s \text{ and } H_s(v_s, 1) = \lambda_s$$

for any $v_s \in V_s$ (for each $s \in S$ the space $\mathbb{T}_s$ is contractible). □

Lemma 3.3. Let S be a nonempty set and let, for each $s \in S$, X_s be a compact space. Let $X = \prod_{s\in S} X_s$. The space X is acyclic if and only if for each $s \in S$, the space X_s is acyclic.

Proof. Assume that X is acyclic. Let $x \in X$, $x = \{x_s\}$ be an arbitrary point and $s_0 \in S$. We want to show that the space X_{s_0} is acyclic. Consider the map

$$X_{s_0} \xrightarrow{h} X \xrightarrow{\pi_{s_0}} X_{s_0},$$

where $h(z_{s_0}) = \{y_s\}$, $y_s = x_s$ for each $s \neq s_0$, $y_{s_0} = z_{s_0}$ and π_{s_0} is a projection. We observe that

$$\pi_{s_0} \circ h = Id_{X_{s_0}}.$$

Hence, the map

$$h_* : H_*(X_{s_0}) \to H_*(X)$$

is a monomorphism, so X_{s_0} is an acyclic space. Assume now that, for each $s \in S$ the space X_s is acyclic. From the mathematical literature (see, [5]) we know that if the spaces $X_1, ..., X_n$ are acyclic, then the space $X_1 \times ... \times X_n$ is acyclic. Let $\Sigma = \{\xi \subset S : \xi$ is a finite set$\}$, then $(\Sigma, \leq)$ is a directed set, where $\leq$ is an inclusion. Given that

$$X = \varprojlim \{Y_\xi, \pi^\xi_\zeta, \Sigma\},$$

where $Y_\xi = X_{s_1} \times X_{s_2} \times ... \times X_{s_n}$, $\xi = \{s_1, s_2, ..., s_n\} \subset S$ and for each $\zeta \leq \xi$, $\pi^\xi_\zeta : Y_\xi \to Y_\zeta$ is a projection. It follows from the continuity of the Čech homology that the space X is acyclic. □

Lemma 3.4. Let Q be a Hilbert cube. Let S be a nonempty set and let $X = \prod_{s\in S} X_s$. Assume that, for each $s \in S$, X_s is a compact subset of $Q_s = Q$. The space X has a trivial shape in $\mathbb{T} = \prod_{s\in S} Q_s$ if and only if for each $s \in S$ a space X_s has a trivial shape in Q_s.

Proposition 3.5. If $X \subset \mathbb{T}$ has a trivial shape in $\mathbb{T}$, then X is acyclic.

Proof. Let

$$\Sigma = \{K : \ K \text{ is a compact neighborhood of } X \text{ in } \mathbb{T}\}$$

a directed set from the inclusion $\leq$, that is, $(\xi \leq \zeta) \Leftrightarrow (K_\zeta \subset K_\xi)$ for each $\xi, \zeta \in \Sigma$ and let

$$\mathbf{X} = \{K_\zeta, \ j^\zeta_\xi, \ \Sigma\}$$

be an inverse system, where for $\xi \leq \zeta$, $j^\zeta_\xi : K_\zeta \to K_\xi$ is an inclusion. Then the inclusion $i_\xi : X \to K_\xi$ is homotopic to a constant map $C_\xi : X \to K_\xi$, $C_\xi(x) = x_0$ for each $x \in X$, $\xi \in \Sigma$ and an arbitrary $x_0 \in X$. Since Čech homology is continuous, we obtain

$$(\varprojlim i_\xi)_* = (\varprojlim C_\xi)_*,$$

where

$$(\varprojlim i_\xi)_* : H_*(X) \to H_*(\varprojlim \mathbf{X})$$

is an isomorphism, so X is acyclic. □

Definition 3.6. ***(Cell-like)*** Let X be a space. A perfect map $\alpha : Z \to X$ is cell-like if for each compact set $K \subset X$ there exist a Tychonoff cube $\mathbb{T}$ and an embedding $h : \alpha^{-1}(K) \to \mathbb{T}$ such that for each $x \in K$ the set $h(\alpha^{-1}(x))$ has a trivial shape in $\mathbb{T}$.

Let Δ be family sets of compact and nonempty spaces such that the following conditions are satisfied:

(3) if A is a single-element space, then $A \in \Delta$.

(4) If, for each $s \in S$, $A_s \in \Delta$ then $\left(\prod_{s\in S} A_s\right) \in \Delta$,

where S is any, nonempty set. For each $A \in \Delta$ there exists a Tychonoff cube $\mathbb{T}$ and an embedding $h : A \to \mathbb{T}$ such that

$$h(A) \in \Delta. \tag{5}$$

If $A \in \Delta$ is a metrizable space then there exists an embedding $h : A \to Q$ such that

$$h(A) \in \Delta \tag{6}$$

where Q is a Hilbert cube. Let X be a space. We will say that a perfect map $\alpha : Z \to X$ is a Δ map if for each compact set $K \subset X$ there exist a Tychonoff cube $\mathbb{T}$ and an embedding $h : \alpha^{-1}(K) \to \mathbb{T}$ such that for each $x \in K$ the set $h(\alpha^{-1}(x)) \in \Delta$. We observe that if $\alpha \in \mathbb{D}(X)$, then for each nonempty set $B \subset X$ (not necessarily compact) $\alpha_{\alpha^{-1}(B)} \in \mathbb{D}(B)$, where $\alpha_{\alpha^{-1}(B)}$ is a restriction of α to the set $\alpha^{-1}(B)$. Denote

$$\mathbb{D}(X) = \{\alpha : Z \to X;\ \alpha \text{ is a } \Delta \text{ map}\}. \tag{7}$$

The following are examples of $\mathbb{D}$ type sets:

$$\mathbb{HOM}(X) = \{\alpha : Z \to X;\ \alpha \text{ is a homeomorphism}\}, \tag{8}$$

$$\mathbb{CELL}(X) = \{\alpha : Z \to X;\ \alpha \text{ is a cell-like map}\}, \tag{9}$$

$$\mathbb{V}(X) = \{\alpha : Z \to X;\ \alpha \text{ is a Vietoris map}\}. \tag{10}$$

We observe that

$$\mathbb{HOM}(X) \subset \mathbb{CELL}(X) \subset \mathbb{V}(X). \tag{11}$$

4 Relative extensions of maps

In this section we will define the notion of relative extension of maps and prove some of its properties.

Definition 4.1. *(see, Definition 2.7)* We say that a space X is a relative extension (relative neighborhood extension) (we write $X \in ESR(\mathbb{D})$, $(X \in NESR(\mathbb{D}))$) if for each compact set $K \subset X$ there exists a space Z_K, $\alpha_K : Z_K \to K$, $\alpha_K \in \mathbb{D}(K)$ such that for each compact space Y, for each closed set $A \subset Y$ and for each continuous map $f : A \to Z_K$ the map $\alpha_K \circ f$ has a continuous extension $F : Y \to X$ $(F : U \to X)$, where $U \subset Y$ is some open set such that $A \subset U$, that is the following diagram:

$$\begin{array}{ccc} Z_K & \xrightarrow{j \circ \alpha_K} & X \\ \uparrow f & & \uparrow F \\ A & \xrightarrow{i} & T, \end{array}$$

is commutative, where $T = Y$ $(T = U)$, $i : A \hookrightarrow T$ and $j : K \hookrightarrow X$ are inclusions.

Proposition 4.2. A space $X \in NESR(\mathbb{HOM})(ESR(\mathbb{HOM}))$ if and only if $X \in NES(ES)$.

Proof. Let $X \in NESR(\mathbb{HOM})$ and let $f : A \to X$ be a continuous map, where $A \subset Y$ is a closed subset of a compact space Y. We denote by $K = f(A)$. There exists a map $\alpha_K : Z_K \to K$, $\alpha_K \in \mathbb{HOM}(K)$ such that the conditions of Definition 4.1 are satisfied. Given that

$$A \xrightarrow{\widetilde{f}} K \xrightarrow{\alpha_K^{-1}} Z_K \xrightarrow{\alpha_K} K \xrightarrow{i} X,$$

where $\widetilde{f}(y) = f(y)$ for each $y \in A$, α_K^{-1} is an inverse homeomorphism and i is an inclusion. There exists an extension $G : U \to X$ of $\alpha_K \circ (\alpha_K^{-1} \circ \widetilde{f}) = \widetilde{f}$, where $U \subset Y$ is some open neighborhood of A. The proof in the opposite direction is obvious and the proof of the second part of this Proposition is analogical. $\square$

Remark 4.3. *(see, [9])* Let X, Y be metrizable spaces. We recall that a space X is a $\mathbb{D}$-retract of Y, if there exist a metrizable space $Z \subset Y$, $\alpha : Z \to X$, $\alpha \in \mathbb{D}(X)$ and $r : Y \to X$ such that $r \circ i = \alpha$, where $i : Z \hookrightarrow Y$ is an inclusion. We will say that the map r is a $\mathbb{D}$-retraction and a space Z is a $\mathbb{D}$-carrier of X in Y. We will write $X \in ANRR(\mathbb{D})$ $(X \in ARR(\mathbb{D}))$ if there exists a normed space E, an open set $V \subset E$ such that X is a $\mathbb{D}$-retract of V (E).

We observe that if $\alpha : X \to Y$ is a perfect (proper) map such that, for each $y \in Y$, the set $\alpha^{-1}(y)$ has a trivial shape then $\alpha \in \mathbb{CELL}(Y)$ (see (6) and Lemma 3.4).

Proposition 4.4. Let X be a metrizable space. If $X \in ANRR(\mathbb{D})(ARR(\mathbb{D}))$ then $X \in NESR(\mathbb{D})(ESR(\mathbb{D}))$.

Proof. Let X be a metrizable space and let $X \in ANRR(\mathbb{D})$. Then, there exists a normed space E, an open set $V \subset E$ such that X is a $\mathbb{D}$-retract of V; that is, there exist a space $Z \subset V$, $\alpha : Z \to X$, $\alpha \in \mathbb{D}(X)$ and $r : V \to X$ such that $r \circ i = \alpha$, where $i : Z \hookrightarrow V$ is an inclusion. Let $K \subset X$ be a compact set, $Z_K = \alpha^{-1}(K)$ and $\alpha_K : Z_K \to K$, $\alpha_K(z) = \alpha(z)$ for each $z \in Z_K$, where $\alpha_K \in \mathbb{D}(K)$. Let us take a compact space Y, a closed set $A \subset Y$ and a continuous map $f : A \to Z_K$. Then, we have

$$A \xrightarrow{f} Z_K \xrightarrow{j} Z \xrightarrow{i} V \xrightarrow{r} X,$$

where j is an inclusion. There exists an extension $G : U \to V$ of $i \circ j \circ f$ (see, Remark 2.6), where $U \subset Y$ is some open neighborhood of A. We define an extension $F : U \to X$ of $\alpha_K \circ f$ by the formula

$$F = r \circ G.$$

The proof of the second part of this Proposition is analogical. $\square$

Proposition 4.5. A space $(X_1 \times X_2) \in NESR(\mathbb{D})(ESR(\mathbb{D}))$ if and only if $X_1 \in NESR(\mathbb{D})(ESR(\mathbb{D}))$ and $X_2 \in NESR(\mathbb{D})(ESR(\mathbb{D}))$.

Proof. Let $(X_1 \times X_2) \in NESR(\mathbb{D})$ and let $K \subset X_1 (K \subset X_2)$ be a compact set. Then a set $K \times \{x_2\} \subset X_1 \times X_2$ $(\{x_1\} \times K \subset X_1 \times X_2)$ is compact, where $(x_1, x_2) \in X_1 \times X_2$ is an arbitrary point. From the assumption there exists a map $\alpha_K : Z_K \to K \times \{x_2\}$

$(\alpha_K : Z_K \to \{x_1\}\times K)$, $\alpha_K \in \mathbb{D}(K\times\{x_2\})$ $(\alpha_K \in \mathbb{D}(\{x_1\}\times K))$ such that the conditions of Definition 4.1 are satisfied. Let T be a compact space and let $f : A \to Z_K$ be a continuous map, where $A \subset T$ is a closed set. There exists an extension $G : U \to X_1\times X_2$ of $\alpha_K \circ f$, where $U \subset T$ is an open neighborhood of A. We define a map $F : U \to X_1$ $(F : U \to X_2)$ by the formula

$$F = \pi_1 \circ G \ \ (F = \pi_2 \circ G),$$

where $\pi_i : X_1 \times X_2 \to X_i$ are projections, $i = 1, 2$. Now, let $X_1 \in NESR(\mathbb{D})$ and $X_2 \in NESR(\mathbb{D})$ and let $K \subset X_1 \times X_2$ be a compact set. We denote by $K_i = \pi_i(K)$, where $\pi_i : X_1 \times X_2 \to X_i$ are projections, $i = 1, 2$. There exists a map $\alpha_{K_i} : Z_{K_i} \to K_i$, $\alpha_{K_i} \in \mathbb{D}(K_i)$, $i = 1, 2$ such that the conditions of Definition 4.1 are satisfied. It's obvious that $K \subset K_1 \times K_2$. Let $\alpha = \alpha_{K_1} \times \alpha_{K_2}$, $Z_K = \alpha^{-1}(K)$ and $\alpha_K : Z_K \to K$, $\alpha_K(z) = \alpha(z)$ for each $z \in Z_K$. Let T be a compact space and let $f : A \to Z_K$ be a continuous map, where $A \subset T$ is a closed set. For $i = 1, 2$ we have

$$A \xrightarrow{\ f\ } Z_K \xrightarrow{\ j\ } Z_{K_1} \times Z_{K_2} \xrightarrow{\ \pi_i'\ } Z_{K_i} \xrightarrow{\ \alpha_{K_i}\ } K_i \xrightarrow{\ j_i\ } X_i,$$

where j, j_i are inclusions and $\pi_i' : Z_{K_1} \times Z_{K_2} \to Z_{K_i}$ are projections, $i = 1, 2$. There exist extensions $G_i : V_i \to X_i$ of $\alpha_{K_i} \circ (\pi_i' \circ j \circ f)$, where $V_i \subset T$ some open neighborhoods of A, $i = 1, 2$. Let $U = V_1 \cap V_2$. We define an extension $F : U \to X_1 \times X_2$ of $\alpha_K \circ f$ by the formula

$$F(t) = (G_1(t), G_2(t)) \quad \text{for each } \ t \in U.$$

The proof of the second part of this Proposition is analogical. □

Proposition 4.6. Let S be a nonempty set and let $X = \prod_{s\in S} X_s$. If $X \in NESR(\mathbb{D})$ then $X_s \in NESR(\mathbb{D})$ for each $s \in S$.

Proof. Assume that $X \in NESR(\mathbb{D})$. Let $s_0 \in S$, $K \subset X_{s_0}$ be a compact set and let $\{x_s\} \in X$ be an arbitrary point. Denote

$$P = \prod_{s\in S} Y_s,$$

where for each $s \neq s_0$, $Y_s = \{x_s\}$ and $Y_{s_0} = K$. There exists a map $\alpha_P \in \mathbb{D}(K)$, $\alpha_P : Z_P \to P$ such that the conditions of Definition 4.1 are satisfied. Let $Z_K = Z_P$ and let $\alpha_K = h \circ \alpha_P$, where $h : P \to K$ is a homeomorphism (restriction of a projection $\pi_{s_0} : X \to X_{s_0}$). Let $f : A \to Z_K$ be a continuous map, where A is a closed subset of a compact space Y. There exists an extension $G : U \to X$ of $\alpha_K \circ f$, where $U \subset Y$ is an open neighborhood of A. We define an extension $F : U \to X_{s_0}$ of $\alpha_K \circ f$ by the formula $F = \pi_{s_0} \circ G$ and the proof is complete. □

Proposition 4.7. Let S be a nonempty set and let $X = \prod_{s\in S} X_s$. A space $X \in ESR(\mathbb{D})$ if and only if $X_s \in ESR(\mathbb{D})$ for each $s \in S$.

Proof. Assume that $X \in ESR(\mathbb{D})$. It follows from Proposition 4.6 that $X_s \in ESR(\mathbb{D})$ for each $s \in S$. Let for each $s \in S$ the space $X_s \in ESR(\mathbb{D})$ and let $K \subset X$ be a compact set. We denote by $K_s = \pi_s(K)$, where $\pi_s : X \to X_s$ is a projection for each

$s \in S$. For each $s \in S$, we take a map $\alpha_{K_s} \in \mathbb{D}(K_s)$, $\alpha_{K_s} : Z_{K_s} \to K_s$ such that the conditions of Definition 4.1 are satisfied. It is obvious that $K \subset \prod_{s\in S} K_s$. Let $\alpha = \prod_{s\in S} \alpha_{K_s} : \prod_{s\in S} Z_{K_s} \to \prod_{s\in S} K_s$, $Z_K = \alpha^{-1}(K)$ and let $\alpha_K : Z_K \to K$ be a restriction of α. It is clear that $\alpha_K \in \mathbb{D}(K)$. Let $f : A \to Z_K$ be a map, where A is a closed subset of a compact space Y and let $\pi_s^1 : Z_K \to Z_{K_s}$ be a projection for each $s \in S$. Then, we have

$$A \xrightarrow{f} Z_K \xrightarrow{\pi_s^1} Z_{K_s} \xrightarrow{\alpha_{K_s}} K_s \xrightarrow{i_s} X_s,$$

where i_s is an inclusion, for each $s \in S$. It follows that for each $s \in S$, there exists an extension $F_s : Y \to X_s$ of $\alpha_{K_s} \circ (\pi_s^1 \circ f)$. We define an extension $F : Y \to X$ of $\alpha_K \circ f$ by the formula:

$$F(y) = \{F_s(y)\} \quad \text{for each } \; y \in Y$$

and the proof is complete. □

Example 4.8. Let X be a metrizable and non-movable space, Q be a Hilbert cube and $p : Q \to X$ be a cell-like map (see, [7]). Let S be a nonempty set ($card(S) > \aleph_0$) and let $Y = \prod_{s\in S} Y_s$, where $Y_s = X$ for each $s \in S$. We define a cell-like map $\alpha : \mathbb{T} \to Y$ by the formula $\alpha = \{p_s\}_{s\in S}$, where $\mathbb{T} = \prod_{s\in S} Q_s$ is a Tychonoff cube and for each $s \in S$, $Q_s = Q$ and $p_s = p$. It is clear that $Y \in ESR(\mathbb{CELL})$ (in particular, $Y \in NESR(\mathbb{CELL})$) is a non-metrizable space. We show that $Y \notin NES$. Assume that $Y \in NES$. Then, from Proposition 4.6, $X \in NES$. Hence X is a neighborhood retract of Q, so $X \in ANR$, but it is a contradiction, since X is a non-movable space (see Remark 2.9).

5 The abstract morphism

The symbol $D(X,Y)$ will denote the set of all mappings of the form

$$X \xleftarrow{p} Z \xrightarrow{q} Y,$$

where $p : Z \to X$ denotes a Vietoris map and $q : Z \to Y$ denotes a continuous map. Each such diagram will be denoted by (p, q). Let $(p_1, q_1) \in D(X,Y)$ and $(p_2, q_2) \in D(Y,T)$. The composition of diagrams (see, [5])

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y \xleftarrow{p_2} Z_2 \xrightarrow{q_2} T,$$

is called a diagram $(p, q) \in D(X,T)$

$$X \xleftarrow{p} Z_1 \triangle_{q_1 p_2} Z_2 \xrightarrow{q} T,$$

$$\text{where } \; Z_1 \triangle_{q_1p_2} Z_2 = \{(z_1, z_2) \in Z_1 \times Z_2 : \; q_1(z_1) = p_2(z_2)\},$$

$$p = p_1 \circ \pi_1, \;\; q = q_2 \circ \pi_2,$$

$$Z_1 \xleftarrow{\pi_1} Z_1 \triangle_{q_1p_2} Z_2 \xrightarrow{\pi_2} Z_2,$$

$$\pi_1(z_1, z_2) = z_1 \text{ (Vietoris map)}, \;\; \pi_2(z_1, z_2) = z_2 \;\text{ for each }\; (z_1, z_2) \in Z.$$

It shall be written

$$(p,q) = (p_2,q_2) \circ (p_1,q_1).$$

From ([5], p. 201, 202) it also results that the composition of the diagrams satisfies the condition:

$$\text{for each } x \in X \;\; q(p^{-1}(x)) = q_2(p_2^{-1}(q_1(p_1^{-1}(x)))). \tag{12}$$

Let $(p_1,q_1),(p_2,q_2) \in D(X,Y)$. Assume that, in the set $D(X,Y)$ we have an equivalency relation (it is denoted as $\sim_a$) such that the following conditions are satisfied (see [14, 13, 8]):

$$((p_1,q_1) \sim_a (p_2,q_2)) \Rightarrow (\text{for each } x \in X \;\; q_1(p_1^{-1}(x)) = q_2(p_2^{-1}(x))), \tag{13}$$

$$((p_1,q_1) \sim_a (p_2,q_2)) \Rightarrow (q_{1*} \circ p_{1*}^{-1} = q_{2*} \circ p_{2*}^{-1}), \tag{14}$$

Let $(p_3,q_3),(p_4,q_4) \in D(Y,T)$.

(15)

$((p_1,q_1) \sim_a (p_2,q_2) \text{ and } (p_3,q_3) \sim_a (p_4,q_4)) \Rightarrow (((p_3,q_3) \circ (p_1,q_1)) \sim_a ((p_4,q_4) \circ (p_2,q_2)))$.

The set $M_a(X,Y) = D(X,Y)_{/\sim_a}$ will be called a set of abstract morphisms (a-morphism). Let $(p,q) \in D(X,Y)$. For any $\varphi_a \in M_a(X,Y)$ the set $\varphi(x) = q(p^{-1}(x))$ where $\varphi_a = [(p,q)]_a$ is called an image of the point x in the a-morphism φ_a. We denote by $\varphi : X \to_a Y$ a multivalued map determined by $\varphi_a \in M_a(X,Y)$. We observe from (12) and (15) that if $\varphi : X \to_a Y$ is determined by $\varphi_a = [(p_1,q_1)]_a$ and $\psi : Y \to_a T$ is determined by $\psi_a = [(p_2,q_2)]_a$ then $\psi \circ \varphi : X \to_a T$ is determined by

$$(\psi \circ \varphi)_a = [((p_2,q_2) \circ (p_1,q_1))]_a.$$

We recall that a multivalued map $\varphi : X \multimap Y$ is acyclic if for each $x \in X$ the set $\varphi(x)$ is compact and acyclic. An acyclic map $\varphi : X \to_a Y$ is determined by $\varphi_a = [(p_\varphi, q_\varphi)]_a$, where

$$X \xleftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} Y$$

are maps given by formulas: $p_\varphi(x,y) = x$, $q_\varphi(x,y) = y$ for each $(x,y) \in \Gamma_\varphi$ and Γ_φ is a graph of φ.

Remark 5.1. Let $\varphi : X \to_a X$ be a map. We observe that a map φ has a fixed point i.e., there exists a point $x \in X$ such that $x \in \varphi(x)$ if and only if for some $(p,q) \in \varphi_a$, p and q have a coincidence point.

Let **TOP** denote categories in which Hausdorff topological spaces are objects and continuous mappings are category mappings. Let $\mathbf{TOP_a}$ denote categories in which Hausdorff topological spaces are objects and multivalued maps determined by abstract morphisms are category mappings. Let $\mathbf{VECT_G}$ denote categories in which linear graded vector spaces are objects and linear mappings of degree zero are category mappings.

Theorem 5.2. *(see [14])* The mapping $\widetilde{\mathbf{H}}_* : \mathbf{TOP_a} \to \mathbf{VECT_G}$ given by the formula

$$\widetilde{\mathbf{H}}_*(\varphi) \equiv \varphi_* = q_* \circ p_*^{-1},$$

where φ is a multivalued map determined by $\varphi_a = [(p,q)]_a$ is a covariant functor and the extension of the functor of the Čech homology $\mathbf{H}_* : \mathbf{TOP} \to \mathbf{VECT_G}$.

Let $\varphi : X \to_a X$ be a map determined by $\varphi_a = [(p,q)]_a$, where $(p,q) \in D(X,X)$. Assume that $\varphi_* = q_* \circ p_*^{-1}$ is a Leray endomorphism (see (14)). Then, we define a Lefschetz number of φ_* by the formula

$$\Lambda(\varphi_*) = \Lambda(q_* \circ p_*^{-1}).$$

We recall that $\varphi : X \to_a X$ is a Lefschetz map if φ_* is a Leray endomorphism and $\Lambda(\varphi_*) \neq 0$ implies that the map φ has a fixed point.

Remark 5.3. A map $\varphi : X \multimap Y$ is admissible if and only if there exists a map $\Delta : X \to_a Y$ such that $\Delta \subset \varphi$, that is, for each $x \in X$, $\Delta(x) \subset \varphi(x)$.

6 The fixed points of compact maps

In this section we will show that the spaces of $NESR(\mathbb{V})$ type has the fixed point property.

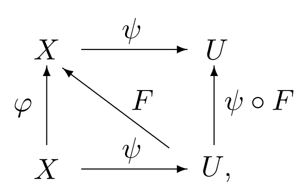
Theorem 6.1. Let $X \in NESR(\mathbb{V})$ and let $\varphi : X \to_a X$ be a compact map. Then φ is a Lefschetz map.

Proof. For $K = \overline{\varphi(X)}$ there exists a map $\alpha_K : Z_K \to K$, $\alpha_K \in \mathbb{V}(K)$ such that the conditions of Definition 4.1 are satisfied. Let $h : Z_K \to \mathbb{T}$ be an embedding and let $S = h(Z_K) \subset \mathbb{T}$, where $\mathbb{T}$ is some Tychonoff cube. We have the following diagrams:

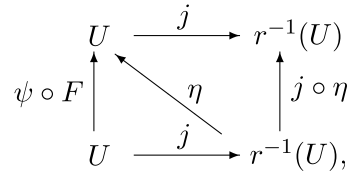
$$S \xrightarrow{h^{-1}} Z_K \xrightarrow{\alpha_K} K \xrightarrow{i_1} X,$$

$$X \xrightarrow{\widetilde{\varphi}} K \xrightarrow{\overleftarrow{\alpha}_K} Z_K \xrightarrow{h} S,$$

where i_1 is an inclusion, h^{-1} is an inverse homeomorphism, $\widetilde{\varphi}(x) = \varphi(x)$ for each $x \in X$ and $\overleftarrow{\alpha}_K(x) = \alpha_K^{-1}(x)$ for each $x \in K$. From the assumption there exists an extension $F : U \to X$ of $\alpha_K \circ h^{-1}$, where $U \subset \mathbb{T}$ is some open set such that $S \subset U$. We get the following commutative diagram:



where $\psi = i_2 \circ h \circ \overleftarrow{\alpha}_K \circ \widetilde{\varphi}$ and $i_2 : S \hookrightarrow U$ is an inclusion . There exists a locally convex space $L(\mathbb{T})$ such that $\mathbb{T}$ is a retract of $L(\mathbb{T})$ (see, [4]). Let $r : L(\mathbb{T}) \to \mathbb{T}$ be a retraction and let $\widetilde{r} : r^{-1}(U) \to U$ be a map given by $\widetilde{r}(x) = r(x)$ for each $x \in r^{-1}(U)$. We have the following commutative diagram:



where j is an inclusion and $\eta = \psi \circ F \circ \widetilde{r}$. From Proposition 2.1 and Theorem 2.4 a Lefschetz number $\Lambda(\varphi_*)$ is well defined and $\Lambda(\varphi_*) = \Lambda((j \circ \eta)_*)$. Assume now, that $\Lambda(\varphi_*) \neq 0$ then from Theorem 2.4 the map $j \circ \eta$ has a fixed point (see, Remark 5.1). Hence, the map $\psi \circ F$ has a fixed point. Let $x \in U$ be a fixed point of $\psi \circ F$. It follows that

$$F(x) \in F(h(\overleftarrow{\alpha}_K(\widetilde{\varphi}(F(x))))) = \widetilde{\varphi}(F(x)) = \varphi(F(x)).$$

Thus, φ is a Lefschetz map. □

7 The fixed points of noncompact maps

Let $\varphi^n \equiv \varphi \circ \varphi \circ ... \circ \varphi$, ($n$th iterate of φ), where $n \in \mathbb{N}$.

Definition 7.1. A map $\varphi : X \to_a X$ is called a compact absorbing contraction (written $\varphi \in CAC(X)$) provided there is an open set $U \subset X$ such that:
(7.1.1) $\varphi(U) \subset U$ and the map $\varphi_U : U \to_a U$, $\varphi_U(x) = \varphi(x)$ for every $x \in X$ is compact,
(7.1.2) for every $x \in X$ there exists $n = n_x$ such that $\varphi^n(x) \subset U$.

Lemma 7.2. *(see [5])* Let $\varphi \in CAC(X)$ and U be an open subset X as in Definition 7.1. If K is a compact subset of X, then there exists $n \in \mathbb{N}$ such that $\varphi^n(K) \subset U$.

Let $\varphi : X \multimap Y$ be a map and let $A \subset X$ and $B \subset Y$ be nonempty sets. Assume that $\varphi(A) \subset B$. We denote by $\widehat{\varphi} : (X, A) \multimap (Y, B)$ a map of pairs, that is, $\widehat{\varphi}(x) = \varphi(x)$ for each $x \in X$.

Lemma 7.3. *(see [5])* Let $\widehat{\varphi} : (X, A) \to_a (X, A)$ be a map of pairs. If any two of endomorphisms $\widehat{\varphi}_* : H(X, A) \to H(X, A)$, $\varphi_* : H(X) \to H(X)$, $\varphi_{A*} : H(A) \to H(A)$ are Leray endomorphisms, then so is the third and

$$\Lambda(\widehat{\varphi}_*) = \Lambda(\varphi_*) - \Lambda(\varphi_{A*}).$$

Lemma 7.4. Let $X \in NESR(\mathbb{D})$. If $U \subset X$ is an open set then $U \in NESR(\mathbb{D})$.

Proof. Let $U \subset X$ be an open set and let $K \subset U$ be a compact set. There exists $\alpha_K : Z_K \to K$, $\alpha_K \in \mathbb{D}(K)$ such that the conditions of Definition 4.1 are satisfied. Let Y be a compact space and let $f : A \to Z_K$ be a continuous map, where $A \subset Y$ is a closed set. From the assumption there exists an extension $F_1 : V_1 \to X$ of $\alpha_K \circ f$, where $V_1 \subset Y$ is some open neighborhood of A. The set $V = F_1^{-1}(U) \subset Y$ is an open neighborhood of A. We define an extension $F : V \to U$ of $\alpha_K \circ f$ given by the formula

$$F(x) = F_1(x) \text{ for each } x \in V.$$

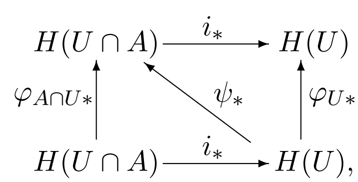
□

Let $\varphi : X \to_a X$ be a map. Denote

$$Fix(\varphi) = \{x \in X : \; x \in \varphi(x)\}.$$

Theorem 7.5. Let X be a space and let $\varphi \in CAC(X)$. Assume further that there exists a space $A \subset X$ such that $A \in NESR(\mathbb{V})$ and $\overline{\varphi(U)} \subset A$, where U is chosen according to Definition 7.1, then φ is a Lefschetz map.

Proof. Let $\psi : U \to_a U \cap A$ be a map given by $\psi(x) = \varphi(x)$ for all $x \in U$. By the assumption, a map ψ is well-defined. We observe that $(U \cap A) \in NESR(\mathbb{V})$ (see, Lemma 7.4). A homomorphism $\widehat{\varphi}_* : H(X, U) \to H(X, U)$ is weakly nilpotent (see, [12]). Hence, from Proposition 2.2 we get $\Lambda(\widehat{\varphi}_*) = 0$. We have a following commutative diagram:



where $i : U \cap A \hookrightarrow U$ is an inclusion. From the above diagram and Proposition 2.1, it results that φ_{U*} is a Leray endomorphism and $\Lambda(\varphi_{U*}) = \Lambda(\varphi_{A\cap U*})$ (see, Theorem 6.1). Hence, from Lemma 7.3, we get that φ_* is a Leray endomorphism and $\Lambda(\varphi_*) = \Lambda(\varphi_{U*})$. Assume that $\Lambda(\varphi_*) \neq 0$. Then $\Lambda(\varphi_{A\cap U*}) \neq 0$ and $\varphi_{A\cap U}$ has a fixed point (see, Theorem 6.1). It is clear that $Fix(\varphi_{A\cap U}) \subset Fix(\varphi)$, so φ is a Lefschetz map. □

8 Conclusion

In section 3 the notions of a trivial shape in topological spaces are given. In section 4 the notions of ES and NES are generalized. Example 4.8 shows that the class of spaces of $NESR(\mathbb{CELL})$ ($ESR(\mathbb{CELL})$) type is essentially wider than the class of spaces of NES (ES) type. We prove that in the class of metrizable spaces $ANRR(\mathbb{D}) \subset NESR(\mathbb{D})$ ($ARR(\mathbb{D}) \subset ESR(\mathbb{D})$). In sections 6 and 7, we prove that the spaces of $NESR(\mathbb{V})$ type (in particular $NESR(\mathbb{CELL})$) have the fixed point property. It is worth mentioning that this article is strongly related to [10, 9, 11].

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