Generalized Fibonacci Numbers: Sum Formulas

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Abstract. In this paper, closed forms of the summation formulas for generalized Fibonacci numbers are presented. As special cases, we give summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. We present the proofs to indicate how these formulas, in general, were discovered. Of course, all the listed formulas may be proved by induction, but that method of proof gives no clue about their discovery.

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1. Introduction

In [4] Horadam defined a generalization of Fibonacci sequence as a second-order linear recurrence sequence $\{W_n(W_0, W_1; r, s)\}$, or simply $\{W_n\}$, as follows:

$$W_n = rW_{n-1} + sW_{n-2}; \quad W_0 = a, \; W_1 = b, \quad (n \geq 2) \tag{1.1}$$

where W_0, W_1 are arbitrary complex numbers and r, s are complex numbers, see also Horadam [3], [5] and [6]. Now these generalized Fibonacci numbers $\{W_n(a, b; r, s)\}$ are also called Horadam numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, \ldots$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences.

Name of sequence	Notation: $W_n(a,b;r,s)$	OEIS: [12]
Fibonacci	$F_n = W_n(0,1;1,1)$	A000045
Lucas	$L_n = W_n(2,1;1,1)$	A000032
Pell	$P_n = W_n(0,1;2,1)$	A000129
Pell-Lucas	$Q_n = W_n(2,2;2,1)$	A002203
Jacobsthal	$J_n = W_n(0,1;1,2)$	A001045
Jacobsthal-Lucas	$j_n = W_n(2,1;1,2)$	A014551

In this work, we investigate some summation formulas of generalized Fibonaci numbers. We present some works on summing formulas of the numbers in the following Table 2.

Table 2. A few special study of sum formulas.

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[7], [9, 10]
Generalized Fibonacci	[1,8,13,14,15]
Generalized Tribonacci	[2,11,16,17]
Generalized Tetranacci	[18,19, 23]
Generalized Pentanacci	[20,21]
Generalized Hexanacci	[22]

2. Summing Formulas of Generalized Fibonacci Numbers with Positive Subscripts

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

THEOREM 2.1. *Let x be a complex number. For $n \geq 0$ we have the following formulas:*

(a): *If $sx^2 + rx - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_k = \frac{x^{n+2}W_{n+2} + x^{n+1}(1-rx)W_{n+1} - xW_1 + (rx-1)W_0}{sx^2 + rx - 1}.$$

(b): *If $r^2x - s^2x^2 + 2sx - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{-x^{n+1}(sx-1)W_{2n+2} + rsx^{n+2}W_{2n+1} - rxW_1 + (r^2x + sx - 1)W_0}{r^2x - s^2x^2 + 2sx - 1}.$$

(c): *If $r^2x - s^2x^2 + 2sx - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{rx^{n+1}W_{2n+2} - sx^{n+1}(sx-1)W_{2n+1} + (sx-1)W_1 - rsxW_0}{r^2x - s^2x^2 + 2sx - 1}.$$

Proof.

(a): Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2}$$

i.e.

$$sW_{n-2} = W_n - rW_{n-1}$$

we obtain

$$\begin{array}{rcl} sx^1W_1 & = & x^1W_3 - rx^1W_2 \\ sx^2W_2 & = & x^2W_4 - rx^2W_3 \\ & \vdots & \\ sx^{n-1}W_{n-1} & = & x^{n-1}W_{n+1} - rx^{n-1}W_n \\ sx^nW_n & = & x^nW_{n+2} - rx^nW_{n+1}. \end{array}$$

If we add the equations by side by, we get

$$\sum_{k=0}^{n} x^k W_k = \frac{x^{n+2}W_{n+2} + x^{n+1}(1-rx)W_{n+1} - xW_1 + (rx-1)W_0}{sx^2 + rx - 1}.$$

(b) and (c): Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2}$$

we obtain

$$\begin{array}{rcl} rx^1W_3 & = & x^1W_4 - sx^1W_2 \\ rx^2W_5 & = & x^2W_6 - sx^2W_4 \\ rx^3W_7 & = & x^3W_8 - sx^3W_6 \\ & \vdots & \\ rx^{n-1}W_{2n-1} & = & x^{n-1}W_{2n} - sx^{n-1}W_{2n-2} \\ rx^nW_{2n+1} & = & x^nW_{2n+2} - sx^nW_{2n}. \end{array}$$

Now, if we add the above equations by side by, we get

$$r(-W_1 + \sum_{k=0}^{n} x^k W_{2k+1}) = (x^n W_{2n+2} - W_2 - x^{-1}W_0 + \sum_{k=0}^{n} x^{k-1} W_{2k}) - s(-W_0 + \sum_{k=0}^{n} x^k W_{2k}). \tag{2.1}$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2}$$

we write the following obvious equations;

$$\begin{aligned}
rx^1W_2 &= x^1W_3 - sx^1W_1 \\
rx^2W_4 &= x^2W_5 - sx^2W_3 \\
rx^3W_6 &= x^3W_7 - sx^3W_5 \\
&\vdots \\
rx^{n-1}W_{2n-2} &= x^{n-1}W_{2n-1} - sx^{n-1}W_{2n-3} \\
rx^nW_{2n} &= x^nW_{2n+1} - sx^nW_{2n-1}
\end{aligned}$$

Now, if we add the above equations by side by, we obtain

$$r(-W_0 + \sum_{k=0}^{n} x^k W_{2k}) = (-W_1 + \sum_{k=0}^{n} x^k W_{2k+1}) - s(-x^{n+1}W_{2n+1} + \sum_{k=0}^{n} x^{k+1} W_{2k+1}). \tag{2.2}$$

Then, solving the system (2.1)-(2.2), the required result of (b) and (c) follow.

2.1. The Case $x = 1$. The case $x = 1$ of Theorem 2.1 is given in [14]. In this subsection, we only consider the case $x = 1, r = 1, s = 2$ and we present a theorem which its proof is different than given in [14] (in fact the formulas given in the following theorem are in different forms than given in [14]).

Observe that setting $x = 1, r = 1, s = 2$ (i.e. for the generalized Jacobsthal case) in Theorem 2.1 (b) and (c) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas. If $r = 1, s = 2$ then we have the following theorem.

THEOREM 2.2. *If $r = 1, s = 2$ then for $n \geq 0$ we have the following formulas:*

(a):

$$\sum_{k=0}^{n} W_k = \frac{1}{2}(W_{n+2} - W_1).$$

(b):

$$\sum_{k=0}^{n} W_{2k} = \frac{1}{3}((n+3)\,W_{2n+2} - 2\,(n+2)\,W_{2n+1} + W_1 - 3W_0).$$

(c):

$$\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}(-\,(n+1)\,W_{2n+2} + 2\,(n+3)\,W_{2n+1} - 2W_1 + 2W_0).$$

Proof.

(a): Take $x = 1, r = 1, s = 2$ in Theorem 2.1 (a).

(b): We use Theorem 2.1 (b). If we set $r = 1, s = 2$ in Theorem 2.1 (b) then we have

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{-x^{n+1}\,(2x-1)\,W_{2n+2} + 2x^{n+2}W_{2n+1} - xW_1 + (3x-1)\,W_0}{-4x^2+5x-1}.$$

For $x=1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\sum_{k=0}^{n} W_{2k} = \left.\frac{\frac{d}{dx}(-x^{n+1}(2x-1)W_{2n+2}+2x^{n+2}W_{2n+1}-xW_1+(3x-1)W_0)}{\frac{d}{dx}(-4x^2+5x-1)}\right|_{x=1}$$
$$= \frac{1}{3}((n+3)W_{2n+2}-2(n+2)W_{2n+1}+W_1-3W_0).$$

(c): We use Theorem 2.1 (c). If we set $r=1, s=2$ in Theorem 2.1 (c) then we have

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{x^{n+1}W_{2n+2}-2x^{n+1}(2x-1)W_{2n+1}+(2x-1)W_1-2xW_0}{-4x^2+5x-1}.$$

For $x=1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we obtain

$$\sum_{k=0}^{n} W_{2k+1} = \left.\frac{\frac{d}{dx}(x^{n+1}W_{2n+2}-2x^{n+1}(2x-1)W_{2n+1}+(2x-1)W_1-2xW_0)}{\frac{d}{dx}(-4x^2+5x-1)}\right|_{x=1}$$
$$= \frac{1}{3}(-(n+1)W_{2n+2}+2(n+3)W_{2n+1}-2W_1+2W_0).$$

Note that different forms of the sum formulas of the above Theorem (b) and (c) are given in [14].

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 2.3. *For $n \geq 0$, Jacobsthal numbers have the following property:*

(a): $\sum_{k=0}^{n} J_k = \frac{1}{2}(J_{n+2}-1)$.
(b): $\sum_{k=0}^{n} J_{2k} = \frac{1}{3}((n+3)J_{2n+2}-2(n+2)J_{2n+1}+1)$.
(c): $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{3}(-(n+1)J_{2n+2}+2(n+3)J_{2n+1}-2)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.4. *For $n \geq 0$, Jacobsthal-Lucas numbers have the following property:*

(a): $\sum_{k=0}^{n} j_k = \frac{1}{2}(j_{n+2}-1)$.
(b): $\sum_{k=0}^{n} j_{2k} = \frac{1}{3}((n+3)j_{2n+2}-2(n+2)j_{2n+1}-5)$.
(c): $\sum_{k=0}^{n} j_{2k+1} = \frac{1}{3}(-(n+1)j_{2n+2}+2(n+3)j_{2n+1}+2)$.

2.2. The Case $x=-1$. We now consider the case $x=-1$ in Theorem 2.1. The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 2.5. *For $n \geq 0$ we have the following formulas:*

(a): *If $s-r-1 \neq 0$ then*

$$\sum_{k=0}^{n} (-1)^k W_k = \frac{(-1)^n W_{n+2}+(-1)^{n+1}(r+1)W_{n+1}+W_1-(r+1)W_0}{s-r-1}.$$

(b): *If* $-r^2-s^2-2s-1$

$$\sum_{k=0}^{n}(-1)^k W_{2k}=\frac{(-1)^{n+1}(s+1)W_{2n+2}+(-1)^n rsW_{2n+1}+rW_1-\left(r^2+s+1\right)W_0}{-r^2-s^2-2s-1}.$$

(c): *If* $-r^2-s^2-2s-1\neq 0$ *then*

$$\sum_{k=0}^{n}(-1)^k W_{2k+1}=\frac{(-1)^{n+1} rW_{2n+2}+(-1)^{n+1} s(s+1)W_{2n+1}-W_1(s+1)+rsW_0}{-r^2-s^2-2s-1}.$$

Taking $r=s=1$ in Theorem 2.5 (a), (b) and (c) we obtain the following proposition.

Proposition 2.6. *If* $r=s=1$ *then for* $n\geq 0$ *we have the following formulas:*

(a): $\sum_{k=0}^{n}(-1)^k W_k=(-1)^{n+1}W_{n+2}+2(-1)^n W_{n+1}+2W_0-W_1$.

(b): $\sum_{k=0}^{n}(-1)^k W_{2k}=\frac{1}{5}(2(-1)^n W_{2n+2}+(-1)^{n+1}W_{2n+1}-W_1+3W_0)$.

(c): $\sum_{k=0}^{n}(-1)^k W_{2k+1}=\frac{1}{5}((-1)^n W_{2n+2}+2(-1)^n W_{2n+1}+2W_1-W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n=F_n$ with $F_0=0, F_1=1$).

Corollary 2.7. *For* $n\geq 0$, *Fibonacci numbers have the following properties:*

(a): $\sum_{k=0}^{n}(-1)^k F_k=(-1)^{n+1}F_{n+2}+2(-1)^n F_{n+1}-1$.

(b): $\sum_{k=0}^{n}(-1)^k F_{2k}=\frac{1}{5}(2(-1)^n F_{2n+2}+(-1)^{n+1}F_{2n+1}-1)$.

(c): $\sum_{k=0}^{n}(-1)^k F_{2k+1}=\frac{1}{5}((-1)^n F_{2n+2}+2(-1)^n F_{2n+1}+2)$.

Taking $W_n=L_n$ with $L_0=2, L_1=1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.8. *For* $n\geq 0$, *Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n}(-1)^k L_k=(-1)^{n+1}L_{n+2}+2(-1)^n L_{n+1}+3$.

(b): $\sum_{k=0}^{n}(-1)^k L_{2k}=\frac{1}{5}(2(-1)^n L_{2n+2}+(-1)^{n+1}L_{2n+1}+5)$.

(c): $\sum_{k=0}^{n}(-1)^k L_{2k+1}=\frac{1}{5}((-1)^n L_{2n+2}+2(-1)^n L_{2n+1})$.

Taking $r=2, s=1$ in Theorem 2.5 (a), (b) and (c) we obtain the following proposition.

Proposition 2.9. *If* $r=2, s=1$ *then for* $n\geq 0$ *we have the following formulas:*

(a): $\sum_{k=0}^{n}(-1)^k W_k=\frac{1}{2}((-1)^{n+1}W_{n+2}+3(-1)^n W_{n+1}-W_1+3W_0)$.

(b): $\sum_{k=0}^{n}(-1)^k W_{2k}=\frac{1}{4}((-1)^n W_{2n+2}+(-1)^{n+1}W_{2n+1}-W_1+3W_0)$.

(c): $\sum_{k=0}^{n}(-1)^k W_{2k+1}=\frac{1}{4}((-1)^n W_{2n+2}+(-1)^n W_{2n+1}+W_1-W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n=P_n$ with $P_0=0, P_1=1$).

Corollary 2.10. *For* $n\geq 0$, *Pell numbers have the following properties:*

(a): $\sum_{k=0}^{n}(-1)^k P_k = \frac{1}{2}((-1)^{n+1} P_{n+2} + 3(-1)^n P_{n+1} - 1)$.

(b): $\sum_{k=0}^{n}(-1)^k P_{2k} = \frac{1}{4}((-1)^n P_{2n+2} + (-1)^{n+1} P_{2n+1} - 1)$.

(c): $\sum_{k=0}^{n}(-1)^k P_{2k+1} = \frac{1}{4}((-1)^n P_{2n+2} + (-1)^n P_{2n+1} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 2.11. *For $n \geq 0$, Pell-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n}(-1)^k Q_k = \frac{1}{2}((-1)^{n+1} Q_{n+2} + 3(-1)^n Q_{n+1} + 4)$.

(b): $\sum_{k=0}^{n}(-1)^k Q_{2k} = \frac{1}{4}((-1)^n Q_{2n+2} + (-1)^{n+1} Q_{2n+1} + 4)$.

(c): $\sum_{k=0}^{n}(-1)^k Q_{2k+1} = \frac{1}{4}((-1)^n Q_{2n+2} + (-1)^n Q_{2n+1})$.

Observe that setting $x = -1, r = 1, s = 2$ (i.e. for the generalized Jacobsthal case) in Theorem 2.1 (a) makes the right hand side of the sum formula to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formula of (a). If $r = 1, s = 2$ then we have the following theorem.

Theorem 2.12. *If $r = 1, s = 2$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n}(-1)^k W_k = \frac{1}{3}((n+2)(-1)^n W_{n+2} + (2n+3)(-1)^{n+1} W_{n+1} + W_1 - W_0)$.

(b): $\sum_{k=0}^{n}(-1)^k W_{2k} = \frac{1}{10}(3(-1)^n W_{2n+2} + 2(-1)^{n+1} W_{2n+1} - W_1 + 4W_0)$.

(c): $\sum_{k=0}^{n}(-1)^k W_{2k+1} = \frac{1}{10}((-1)^n W_{2n+2} + 6(-1)^n W_{2n+1} + 3W_1 - 2W_0)$.

Proof.

(a): We use Theorem 2.1 (a). If we set $r = 1, s = 2$ in Theorem 2.1 (a) then we have

$$\sum_{k=0}^{n} x^k W_k = \frac{x^{n+2} W_{n+2} - x^{n+1}(x-1) W_{n+1} - xW_1 + (x-1) W_0}{2x^2 + x - 1}.$$

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule.

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k W_k &= \left.\frac{\frac{d}{dx}(x^{n+2} W_{n+2} - x^{n+1}(x-1) W_{n+1} - xW_1 + (x-1) W_0)}{\frac{d}{dx}(2x^2 + x - 1)}\right|_{x=-1} \\ &= \frac{1}{3}((n+2)(-1)^n W_{n+2} + (2n+3)(-1)^{n+1} W_{n+1} + W_1 - W_0).\end{aligned}$$

(b): Taking $x = -1, r = 1, s = 2$ in Theorem 2.1 (b) we obtain (b).

(c): Taking $x = -1, r = 1, s = 2$ in Theorem 2.1 (c) we obtain (c).

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 2.13. *For $n \geq 0$, Jacobsthal numbers have the following property:*

(a): $\sum_{k=0}^{n}(-1)^k J_k = \frac{1}{3}((n+2)(-1)^n J_{n+2} + (2n+3)(-1)^{n+1} J_{n+1} + 1)$.

(b): $\sum_{k=0}^{n}(-1)^k J_{2k} = \frac{1}{10}(3\,(-1)^n\, J_{2n+2} + 2\,(-1)^{n+1}\, J_{2n+1} - 1)$.

(c): $\sum_{k=0}^{n}(-1)^k J_{2k+1} = \frac{1}{10}((-1)^n\, J_{2n+2} + 6\,(-1)^n\, J_{2n+1} + 3)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.14. *For $n \geq 0$, Jacobsthal-Lucas numbers have the following property:*

(a): $\sum_{k=0}^{n}(-1)^k j_k = \frac{1}{3}((n+2)\,(-1)^n j_{n+2} + (2n+3)\,(-1)^{n+1} j_{n+1} - 1)$.

(b): $\sum_{k=0}^{n}(-1)^k j_{2k} = \frac{1}{10}(3\,(-1)^n\, j_{2n+2} + 2\,(-1)^{n+1}\, j_{2n+1} + 7)$.

(c): $\sum_{k=0}^{n}(-1)^k j_{2k+1} = \frac{1}{10}((-1)^n\, j_{2n+2} + 6\,(-1)^n\, j_{2n+1} - 1)$.

2.3. The Case $x = 1+i$. We now consider the complex case $x = 1+i$ in Theorem 2.1. The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 2.15. *For $n \geq 0$ we have the following formulas:*

(a): *If $(1+i)\,r + 2is - 1 \neq 0$, then*

$$\sum_{k=0}^{n}(1+i)^k W_k = \frac{(1+i)^{n+2}\, W_{n+2} - (1+i)^{n+1}\,((1+i)\,r - 1)\, W_{n+1} - (1+i)\, W_1 + ((1+i)\,r - 1)\, W_0}{(1+i)\,r + 2is - 1}.$$

(b): *If $(1+i)\,r^2 - 2is^2 + (2+2i)\,s - 1 \neq 0$ then*

$$\sum_{k=0}^{n}(1+i)^k W_{2k}$$

$$= \frac{-((1+i))^{n+1}\,((1+i)\,s - 1)\, W_{2n+2} + ((1+i))^{n+2}\, rsW_{2n+1} - (1+i)\, rW_1 + \big((1+i)\,r^2 + (1+i)\,s - 1\big)\, W_0}{(1+i)\,r^2 - 2is^2 + (2+2i)\,s - 1}.$$

(c): *If $(1+i)\,r^2 - 2is^2 + (2+2i)\,s - 1 \neq 0$ then*

$$\sum_{k=0}^{n}(1+i)^k W_{2k+1} = \frac{((1+i))^{n+1}\, rW_{2n+2} - ((1+i))^{n+1}\, s\,((1+i)\,s - 1)\, W_{2n+1} + ((1+i)\,s - 1)\, W_1 - (1+i)\, rsW_0}{(1+i)\,r^2 - 2is^2 + (2+2i)\,s - 1}.$$

Taking $r = 1$, $s = 1$ in the last theorem we obtain the following proposition.

Proposition 2.16. *If $r = s = 1$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n}(1+i)^k W_k = \frac{1}{3i}((1+i)^{n+2}\, W_{n+2} - i\,(1+i)^{n+1}\, W_{n+1} - (1+i)\, W_1 + iW_0)$.

(b): $\sum_{k=0}^{n}(1+i)^k W_{2k} = \frac{1}{2+i}(-i\,(1+i)^{n+1}\, W_{2n+2} + (1+i)^{n+2}\, W_{2n+1} - (1+i)\, W_1 + (1+2i)\, W_0)$.

(c): $\sum_{k=0}^{n}(1+i)^k W_{2k+1} = \frac{1}{2+i}((1+i)^{n+1}\, W_{2n+2} - i\,(1+i)^{n+1}\, W_{2n+1} + iW_1 - (1+i)\, W_0)$

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 2.17. *For $n \geq 0$, Fibonacci numbers have the following properties.*

(a): $\sum_{k=0}^{n}(1+i)^k F_k = \frac{1}{3i}((1+i)^{n+2}\, F_{n+2} - i\,(1+i)^{n+1}\, F_{n+1} - 1 - i)$.

(b): $\sum_{k=0}^{n}(1+i)^k F_{2k} = \frac{1}{2+i}(-i\,(1+i)^{n+1}\, F_{2n+2} + (1+i)^{n+2}\, F_{2n+1} - 1 - i)$.

(c): $\sum_{k=0}^{n}(1+i)^k F_{2k+1} = \frac{1}{2+i}((1+i)^{n+1} F_{2n+2} - i(1+i)^{n+1} F_{2n+1} + i)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.18. *For $n \geq 0$, Lucas numbers have the following properties.*

(a): $\sum_{k=0}^{n}(1+i)^k L_k = \frac{1}{3i}((1+i)^{n+2} L_{n+2} - i(1+i)^{n+1} L_{n+1} - 1 + i)$.
(b): $\sum_{k=0}^{n}(1+i)^k L_{2k} = \frac{1}{2+i}(-i(1+i)^{n+1} L_{2n+2} + (1+i)^{n+2} L_{2n+1} + 1 + 3i)$.
(c): $\sum_{k=0}^{n}(1+i)^k L_{2k+1} = \frac{1}{2+i}((1+i)^{n+1} L_{2n+2} - i(1+i)^{n+1} L_{2n+1} - 2 - i)$.

3. Summing Formulas of Generalized Fibonacci Numbers with Negative Subscripts

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.1. *Let x be a complex number. For $n \geq 1$ we have the following formulas:*

(a): *If $s + rx - x^2 \neq 0$, then*

$$\sum_{k=1}^{n} x^k W_{-k} = \frac{-x^{n+1}(s+rx) W_{-n-1} - s x^{n+2} W_{-n-2} + x W_1 + x(x-r) W_0}{s + rx - x^2}.$$

(b): *If $r^2 x + 2sx - s^2 - x^2 \neq 0$ then*

$$\sum_{k=1}^{n} x^k W_{-2k} = \frac{x^{n+1}(s-x) W_{-2n} - rsx^{n+1} W_{-2n-1} + rx W_1 + x(x-s-r^2) W_0}{r^2 x + 2sx - s^2 - x^2}.$$

(c): *If $r^2 x + 2sx - s^2 - x^2 \neq 0$ then*

$$\sum_{k=1}^{n} x^k W_{-2k+1} = \frac{-rx^{n+2} W_{-2n} + s x^{n+1}(s-x) W_{-2n-1} + x(x-s) W_1 + rsx W_0}{r^2 x + 2sx - s^2 - x^2}.$$

Proof. The proof of the theorem can be given as in the proof of Theorem 2.1, so we omit it.

3.1. The Case $x = 1$. The case $x = 1$ of Theorem 3.1 is given in [15], see also [14]. In this subsection, we only consider the case $x = 1, r = 1, s = 2$ and we present a theorem which its proof is different than given in [15] (in fact the formulas given in the following theorem are in different forms than given in [15]).

Observe that setting $x = 1, r = 1, s = 2$ (i.e. for the generalized Jacobsthal case) in Theorem (b) and (c) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas. If $r = 1, s = 2$ then we have the following theorem.

Theorem 3.2. *If $r = 1, s = 2$ then for $n \geq 1$ we have the following formulas:*

(a):

$$\sum_{k=1}^{n} W_{-k} = \frac{1}{2}(-3W_{-n-1} - 2W_{-n-2} + W_1).$$

(b):

$$\sum_{k=1}^{n} W_{-2k} = \frac{1}{3}(nW_{-2n} - 2(n+1)W_{-2n-1} + W_1 - W_0).$$

(c):

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{3}(-(n+2)W_{-2n} + 2nW_{-2n-1} + 2W_0).$$

Proof.

(a): Take $x = 1, r = 1, s = 2$ in Theorem 3.1 (a).

(b): We use Theorem 3.1 (b). If we set $r = 1, s = 2$ in Theorem 3.1 (b) then we have

$$\sum_{k=1}^{n} x^k W_{-k} = \frac{-(x-2)x^{n+1}W_{-2n} - 2x^{n+1}W_{-2n-1} + xW_1 + x(x-3)W_0}{-x^2 + 5x - 4}.$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule.

$$\begin{aligned}\sum_{k=0}^{n} W_{2k} &= \left.\frac{\frac{d}{dx}(-(x-2)x^{n+1}W_{-2n} - 2x^{n+1}W_{-2n-1} + xW_1 + x(x-3)W_0)}{\frac{d}{dx}(-x^2+5x-4)}\right|_{x=1} \\ &= \frac{1}{3}(nW_{-2n} - 2(n+1)W_{-2n-1} + W_1 - W_0).\end{aligned}$$

(c): We use Theorem 3.1 (c). If we set $r = 1, s = 2$ in Theorem 3.1 (c) then we have

$$\sum_{k=1}^{n} x^k W_{-2k+1} = \frac{-x^{n+2}W_{-2n} - 2(x-2)x^{n+1}W_{-2n-1} + x(x-2)W_1 + 2xW_0}{-x^2 + 5x - 4}.$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule.

$$\begin{aligned}\sum_{k=0}^{n} W_{2k+1} &= \left.\frac{\frac{d}{dx}(-x^{n+2}W_{-2n} - 2(x-2)x^{n+1}W_{-2n-1} + x(x-2)W_1 + 2xW_0)}{\frac{d}{dx}(-x^2+5x-4)}\right|_{x=1} \\ &= \frac{1}{3}(-(n+2)W_{-2n} + 2nW_{-2n-1} + 2W_0).\end{aligned}$$

Note that different forms of the sum formulas of the above Theorem (b) and (c) are given in [15].

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

COROLLARY 3.3. *For $n \geq 1$, Jacobsthal numbers have the following property:*

(a): $\sum_{k=1}^{n} J_{-k} = \frac{1}{2}(-3J_{-n-1} - 2J_{-n-2} + 1)$.

(b): $\sum_{k=1}^{n} J_{-2k} = \frac{1}{3}(nJ_{-2n} - 2(n+1)J_{-2n-1} + 1)$.

(c): $\sum_{k=1}^{n} J_{-2k+1} = \frac{1}{3}(-(n+2)J_{-2n} + 2nJ_{-2n-1})$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

COROLLARY 3.4. *For $n \geq 1$, Jacobsthal numbers have the following property:*

(a): $\sum_{k=1}^{n} j_{-k} = \frac{1}{2}(-3j_{-n-1} - 2j_{-n-2} + 1)$.

(b): $\sum_{k=1}^{n} j_{-2k} = \frac{1}{3}(nj_{-2n} - 2(n+1)j_{-2n-1} - 1)$.

(c): $\sum_{k=1}^{n} j_{-2k+1} = \frac{1}{3}(-(n+2)\,j_{-2n} + 2nj_{-2n-1} + 4)$.

3.2. The Case $x = -1$. We now consider the case $x = -1$ in Theorem 3.1. The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.5. *For $n \geq 1$ we have the following formulas:*

(a): *If $r + s - 1 \neq 0$, then*

$$\sum_{k=1}^{n}(-1)^k W_{-k} = \frac{(-1)^{n+1}(r-s)\,W_{-n-1} - (-1)^n sW_{-n-2} - W_1 + (r+1)\,W_0}{s - r - 1}.$$

(b): *If $-r^2 - s^2 - 2s - 1 \neq 0$ then*

$$\sum_{k=1}^{n}(-1)^k W_{-2k} = \frac{(-1)^{n+1}(s+1)\,W_{-2n} + (-1)^n rsW_{-2n-1} - rW_1 + \left(r^2 + s + 1\right) W_0}{-r^2 - s^2 - 2s - 1}.$$

(c): *If $-r^2 - s^2 - 2s - 1 \neq 0$ then*

$$\sum_{k=1}^{n}(-1)^k W_{-2k+1} = \frac{(-1)^{n+1} rW_{-2n} + (-1)^{n+1} s\,(s+1)\,W_{-2n-1} + (s+1)\,W_1 - rsW_0}{-r^2 - s^2 - 2s - 1}.$$

Taking $r = s = 1$ in Theorem 3.5 (a), (b) and (c) we obtain the following proposition.

Proposition 3.6. *If $r = s = 1$ then for $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=1}^{n}(-1)^k W_{-k} = (-1)^n W_{-n-2} + W_1 - 2W_0$.

(b): $\sum_{k=1}^{n}(-1)^k W_{-2k} = \frac{1}{5}(2(-1)^n W_{-2n} - (-1)^n W_{-2n-1} + W_1 - 3W_0)$.

(c): $\sum_{k=1}^{n}(-1)^k W_{-2k+1} = \frac{1}{5}((-1)^n W_{-2n} + 2(-1)^n W_{-2n-1} - 2W_1 + W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 3.7. *For $n \geq 1$, Fibonacci numbers have the following properties.*

(a): $\sum_{k=1}^{n}(-1)^k F_{-k} = (-1)^n F_{-n-2} + 1$.

(b): $\sum_{k=1}^{n}(-1)^k F_{-2k} = \frac{1}{5}(2(-1)^n F_{-2n} - (-1)^n F_{-2n-1} + 1)$.

(c): $\sum_{k=1}^{n}(-1)^k F_{-2k+1} = \frac{1}{5}((-1)^n F_{-2n} + 2(-1)^n F_{-2n-1} - 2)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 3.8. *For $n \geq 1$, Lucas numbers have the following properties.*

(a): $\sum_{k=1}^{n}(-1)^k L_{-k} = (-1)^n L_{-n-2} - 3$.

(b): $\sum_{k=1}^{n}(-1)^k L_{-2k} = \frac{1}{5}(2(-1)^n L_{-2n} - (-1)^n L_{-2n-1} - 5)$.

(c): $\sum_{k=1}^{n}(-1)^k L_{-2k+1} = \frac{1}{5}((-1)^n L_{-2n} + 2(-1)^n L_{-2n-1})$.

Taking $r=2, s=1$ in Theorem 3.5 (a), (b) and (c) we obtain the following proposition.

PROPOSITION 3.9. *If $r=2, s=1$ then for $n\geq 1$ we have the following formulas:*

(a): $\sum_{k=1}^{n}(-1)^{k}W_{-k}=\frac{1}{2}((-1)^{n}W_{-n-1}+(-1)^{n}W_{-n-2}+W_{1}-3W_{0})$.
(b): $\sum_{k=1}^{n}(-1)^{k}W_{-2k}=\frac{1}{4}((-1)^{n}W_{-2n}+(-1)^{n+1}W_{-2n-1}+W_{1}-3W_{0})$.
(c): $\sum_{k=1}^{n}(-1)^{k}W_{-2k+1}=\frac{1}{4}((-1)^{n}W_{-2n}+(-1)^{n}W_{-2n-1}-W_{1}+W_{0})$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n=P_n$ with $P_0=0, P_1=1$).

COROLLARY 3.10. *For $n\geq 1$, Pell numbers have the following properties.*

(a): $\sum_{k=1}^{n}(-1)^{k}P_{-k}=\frac{1}{2}((-1)^{n}P_{-n-1}+(-1)^{n}P_{-n-2}+1)$.
(b): $\sum_{k=1}^{n}(-1)^{k}P_{-2k}=\frac{1}{4}((-1)^{n}P_{-2n}+(-1)^{n+1}P_{-2n-1}+1)$.
(c): $\sum_{k=1}^{n}(-1)^{k}P_{-2k+1}=\frac{1}{4}((-1)^{n}P_{-2n}+(-1)^{n}P_{-2n-1}-1)$.

Taking $W_n=Q_n$ with $Q_0=2, Q_1=2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

COROLLARY 3.11. *For $n\geq 1$, Pell-Lucas numbers have the following properties.*

(a): $\sum_{k=1}^{n}(-1)^{k}Q_{-k}=\frac{1}{2}((-1)^{n}Q_{-n-1}+(-1)^{n}Q_{-n-2}-4)$.
(b): $\sum_{k=1}^{n}(-1)^{k}Q_{-2k}=\frac{1}{4}((-1)^{n}Q_{-2n}+(-1)^{n+1}Q_{-2n-1}-4)$.
(c): $\sum_{k=1}^{n}(-1)^{k}Q_{-2k+1}=\frac{1}{4}((-1)^{n}Q_{-2n}+(-1)^{n}Q_{-2n-1})$.

Observe that setting $x=-1, r=1, s=2$ (i.e. for the generalized Jacobsthal case) in Theorem 3.1 (a) makes the right hand side of the sum formula to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formula of (a). If $r=1, s=2$ then we have the following theorem.

THEOREM 3.12. *If $r=1, s=2$ then for $n\geq 1$ we have the following formulas:*

(a): $\sum_{k=1}^{n}(-1)^{k}W_{-k}=\frac{1}{3}(n(-1)^{n+1}W_{-n-1}+2\left(n+2\right)(-1)^{n}W_{-n-2}+W_{1}-3W_{0})$.
(b): $\sum_{k=1}^{n}(-1)^{k}W_{-2k}=\frac{1}{10}(3\left(-1\right)^{n}W_{-2n}+2\left(-1\right)^{n+1}W_{-2n-1}+W_{1}-4W_{0})$.
(c): $\sum_{k=1}^{n}(-1)^{k}W_{-2k+1}=\frac{1}{10}((-1)^{n}W_{-2n}+6\left(-1\right)^{n}W_{-2n-1}-3W_{1}+2W_{0})$.

Proof.

(a): We use Theorem 3.1 (a). If we set $r=1, s=2$ in Theorem 3.1 (a) then we have

$$\sum_{k=1}^{n}x^{k}W_{-k}=\frac{-\left(x+2\right)x^{n+1}W_{-n-1}-2x^{n+2}W_{-n-2}+xW_{1}+x\left(x-1\right)W_{0}}{-x^{2}+x+2}.$$

For $x=-1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule.

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k}W_{-k} &= \left.\frac{\frac{d}{dx}(-(x+2)x^{n+1}W_{-n-1}-2x^{n+2}W_{-n-2}+xW_{1}+x(x-1)W_{0})}{\frac{d}{dx}(-x^{2}+x+2)}\right|_{x=-1} \\ &= \frac{1}{3}(n(-1)^{n+1}W_{-n-1}+2(n+2)(-1)^{n}W_{-n-2}+W_{1}-3W_{0}).\end{aligned}$$

(b): Take $x=-1,r=1,s=2$ in Theorem 3.1 (b).

(c): Take $x=-1,r=1,s=2$ in Theorem 3.1 (c).

From the last theorem, we have the following corollary which gives sum formula of Jacobsthal numbers (take $W_n=J_n$ with $J_0=0,J_1=1$).

Corollary 3.13. *For $n\geq 1$, Jacobsthal numbers have the following property:*

(a): $\sum_{k=1}^{n}(-1)^{k}J_{-k}=\frac{1}{3}(n(-1)^{n+1}J_{-n-1}+2(n+2)(-1)^{n}J_{-n-2}+1)$.

(b): $\sum_{k=1}^{n}(-1)^{k}J_{-2k}=\frac{1}{10}(3(-1)^{n}J_{-2n}+2(-1)^{n+1}J_{-2n-1}+1)$.

(c): $\sum_{k=1}^{n}(-1)^{k}J_{-2k+1}=\frac{1}{10}((-1)^{n}J_{-2n}+6(-1)^{n}J_{-2n-1}-3)$.

Taking $W_n=j_n$ with $j_0=2,j_1=1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.14. *For $n\geq 1$, Jacobsthal-Lucas numbers have the following property:*

(a): $\sum_{k=1}^{n}(-1)^{k}j_{-k}=\frac{1}{3}(n(-1)^{n+1}j_{-n-1}+2(n+2)(-1)^{n}j_{-n-2}-5)$.

(b): $\sum_{k=1}^{n}(-1)^{k}j_{-2k}=\frac{1}{10}(3(-1)^{n}j_{-2n}+2(-1)^{n+1}j_{-2n-1}-7)$.

(c): $\sum_{k=1}^{n}(-1)^{k}j_{-2k+1}=\frac{1}{10}((-1)^{n}j_{-2n}+6(-1)^{n}j_{-2n-1}+1)$.

3.3. The Case $x=1+i$. We now consider the complex case $x=1+i$ in Theorem 3.1. The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.15. *For $n\geq 1$ we have the following formulas:*

(a): *If $(1+i)r+s-2i\neq 0$, then*

$$\sum_{k=1}^{n}(1+i)^{k}W_{-k}=\frac{-(1+i)^{n+1}((1+i)r+s)W_{-n-1}-(1+i)^{n+2}sW_{-n-2}+(1+i)W_{1}-(1+i)(r-1-i)W_{0}}{(1+i)r+s-2i}.$$

(b): *If $(1+i)r^{2}-s^{2}+(2+2i)s-2i\neq 0$ then*

$$\sum_{k=1}^{n}(1+i)^{k}W_{-2k}=\frac{(1+i)^{n+1}(s-1-i)W_{-2n}-(1+i)^{n+1}rsW_{-2n-1}+(1+i)rW_{1}-(1+i)\left(r^{2}+s-1-i\right)W_{0}}{(1+i)r^{2}-s^{2}+(2+2i)s-2i}.$$

(c): *If $(1+i)r^{2}-s^{2}+(2+2i)s-2i\neq 0$ then*

$$\sum_{k=1}^{n}(1+i)^{k}W_{-2k+1}=\frac{-(1+i)^{n+2}rW_{-2n}+(1+i)^{n+1}s(s-1-i)W_{-2n-1}-(1+i)(s-1-i)W_{1}+(1+i)rsW_{0}}{(1+i)r^{2}-s^{2}+(2+2i)s-2i}.$$

Taking $r=s=1$ in the last theorem we obtain the following proposition.

PROPOSITION 3.16. *If $r = s = 1$ then for $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=1}^{n}(1+i)^k W_{-k} = \frac{1}{2-i}(-(2+i)(1+i)^{n+1} W_{-n-1} - (1+i)^{n+2} W_{-n-2} + (1+i) W_1 - (1-i) W_0)$.

(b): $\sum_{k=1}^{n}(1+i)^k W_{-2k} = \frac{1}{2+i}(-i(1+i)^{n+1} W_{-2n} - (1+i)^{n+1} W_{-2n-1} + (1+i) W_1 - 2W_0)$.

(c): $\sum_{k=1}^{n}(1+i)^k W_{-2k+1} = \frac{1}{2+i}(-(1+i)^{n+2} W_{-2n} - i(1+i)^{n+1} W_{-2n-1} - (1-i) W_1 + (1+i) W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

COROLLARY 3.17. *For $n \geq 1$, Fibonacci numbers have the following properties.*

(a): $\sum_{k=1}^{n}(1+i)^k F_{-k} = \frac{1}{2-i}(-(2+i)(1+i)^{n+1} F_{-n-1} - (1+i)^{n+2} F_{-n-2} + 1 + i)$.

(b): $\sum_{k=1}^{n}(1+i)^k F_{-2k} = \frac{1}{2+i}(-i(1+i)^{n+1} F_{-2n} - (1+i)^{n+1} F_{-2n-1} + 1 + i)$.

(c): $\sum_{k=1}^{n}(1+i)^k F_{-2k+1} = \frac{1}{2+i}(-(1+i)^{n+2} F_{-2n} - i(1+i)^{n+1} F_{-2n-1} - 1 + i)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

COROLLARY 3.18. *For $n \geq 1$, Lucas numbers have the following properties.*

(a): $\sum_{k=1}^{n}(1+i)^k L_{-k} = \frac{1}{2-i}(-(2+i)(1+i)^{n+1} L_{-n-1} - (1+i)^{n+2} L_{-n-2} - 1 + 3i)$.

(b): $\sum_{k=1}^{n}(1+i)^k L_{-2k} = \frac{1}{2+i}(-i(1+i)^{n+1} L_{-2n} - (1+i)^{n+1} L_{-2n-1} - 3 + i)$.

(c): $\sum_{k=1}^{n}(1+i)^k L_{-2k+1} = \frac{1}{2+i}(-(1+i)^{n+2} L_{-2n} - i(1+i)^{n+1} L_{-2n-1} + 1 + 3i)$.

4. Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

We can summarize the sections as follows:

- In section 1, we present some background about generalized Fibonacci numbers.
- In section 2, summation formulas have been presented for the Fibonacci numbers with positive subscripts. As special cases, summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers with positive subscripts have been given.
- In section 3, summation formulas have been presented for the Fibonacci numbers with negative subscripts. As special cases, summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers with negative subscripts have been given.

Competing Interests

Author have declared that no competing interests exist.

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