Oscillation criteria for a class of third-order differential equations with neutral term

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ABSTRACT

This paper is concerned with oscillation criteria for a class of third-order differential equations with neutral term by using some necessary analysis techniques, some sufficient conditions for oscillation are obtained, some examples are provided to illustrate the main results.

Keywords: Oscillation; Neutral; Third-order; Differential equations

1 Introduction

In this paper, we consider the oscillatory and asymptotic properties for a class of third-order nonlinear differential equation with damped term

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}\left[x(t)+a(t)x(\mu(t))\right]'\right)'\right)'+q(t)f(x(\delta(t)))=0,\quad t\geq t_0 \tag{E}$$

As usual, we use the notation, $u(t)=x(t)+a(t)x(\mu(t))$. In what follows, it is always assume

(C1) $p(t), r(t), a(t), q(t), \delta(t), \mu(t) \in C\left([t_0,\infty),(0,\infty)\right)$,

(C2) $\int_{t_0}^{\infty} p(t)\mathrm{d}t=\int_{t_0}^{\infty} r(t)\mathrm{d}t=\infty$, $r'(t)>0$,

(C3) $\mu(t)\leq t$, $\lim_{t\to\infty}\mu(t)=\lim_{t\to\infty}\delta(t)=\infty$,

(C4) $0\leq a(t)\leq a_0<1$, $f\in C(R,R)$, $f'(v)>0$, $\frac{f(v)}{v}\geq\lambda$, for all $v\neq 0$, and for some $\lambda>0$.

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By a solution of equation (E) we mean a continuous function $x(t)$ definned on an interval $[t_0,\infty)$ such that $\left(\frac{1}{r(t)}\left[x(t)+a(t)x(\mu(t))\right]'\right)'$ is continuously differentiable satifies (E), we assum that equation (E) have such solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[t_0,\infty)$, otherwise, it is called nonoscillatory. We say equation (E) is oscillatory if all its continuable solutions are oscillatory.

In what follows, we consider only proper solution of the equation (E) which are defined for all large t. More and more people are interested in oscillatory and nonoscillatory criteria to be shown[1−7]. Our principal goal in this paper is to derive new oscillation criteria for equation (E), without requiring restrictive condition (4) and (7) in [1]

For simplicity, we introduce the following nonation:

$$u^{[0]}(t)=u(t),\quad u^{[1]}(t)=\frac{1}{r(t)}u'(t),\quad u^{[2]}(t)=\frac{1}{p(t)}(u^{[1]}(t))'$$

lemma 1. Let $x(t)$ be a nonscillatory solutionof (E), then there exists a T_x for $t>T_x\geq t_0$, such that $u(t)$ has only the following two cases.

(i) $u(t)u^{[1]}(t)<0,\quad u(t)u^{[2]}(t)>0,$

(ii) $u(t)u^{[1]}(t)>0,\quad u(t)u^{[2]}(t)>0$.

Proof. Without loss of generality we may assume that $x(t)$ is eventually positive, i.e. there exists $T_x\geq t_0$ such that $x(t)>0$, $u(t)>0$ for $t\geq T_x$. (If it is an eventually negative ,the proof is similar). Using (E) we get $(u^{[2]}(t))'<0$, eventually. Then $u^{[2]}(t)$ is decreasing and of one sign for $t\geq T_x$. If we admit $u^{[2]}(t)<0$, then there exists aconstant $M>0$ such that

$$\frac{1}{p(t)}u^{[1]}(t))'\leq -M<0,$$

$$(u^{[1]}(t))'\leq -Mp(t)$$

Integrating from T_x to t, we obtain

$$u^{[1]}(t)\leq u^{[1]}(T_x)-M\int_{T_x}^{t}p(s)\mathrm{d}s$$

Leting $t\to\infty$ and using (C2), we get $u^{[1]}(t)<0$, which together with $r'(t)>0$ and $u^{[2]}(t)=\frac{r(t)u''(t)-r'(t)u'(t)}{p(t)r^2(t)}<0$.

We get $u''(t)<0$, from $u^{[1]}(t)=\frac{1}{r(t)}u'(t)<0$, we obtain $u'(t)<0$, this implies $u(t)<0$. This constradiction shows that $u^{[2]}(t)>0$, thus either $u^{[1]}(t)<0$ or $u^{[1]}(t)>0$ holds, eventually. The proof is completed.

lemma 2. Assume that x is a solution of (E), $u(t)$ has the proper **(ii)**, then

$$(1-a_0)|u(t)|\leq |x(t)|\leq |u(t)|, \tag{1.1}$$

for $t \geq T$ and

$$\lim_{t\to\infty} |u(t)| = \lim_{t\to\infty} |x(t)| = \infty \tag{1.2}$$

The proof of this Lemma is similar to Lemma 1 of refference of [1], and hence is ommitted.

2 Oscillation theorems

Theorem 1. Assume that

$$\int_{t_1}^{\infty} r(w) \int_{w}^{\infty} p(v) \int_{v}^{\infty} q(s)\mathrm{d}s\mathrm{d}v\mathrm{d}w = \infty \tag{2.1}$$

Moreover, assume that $\delta(t) < t$ and there exists function $g(t)$ such that

$$g(t) \in C\left([t_0, \infty), R\right), \quad g(t) > t, \ \delta\left(g(t)\right) \leq t,$$

and

$$lim_{t\to\infty} \int_{t}^{g(t)} q(s) \int_{t_0}^{\delta(s)} r(v) \int_{t_0}^{v} p(w)\mathrm{d}w\mathrm{d}v\mathrm{d}s = \infty \tag{2.2}$$

Then any proper solution x of (E) is either oscillatory or satisfies $lim_{t\to\infty} x(t) = 0$.

Proof. Without loss of generality we may assume that x is an eventually positive solution, we first assume that $u(t)$ has the proper **(i)**. Then there exists $T_x \geq t_0$ such that $u(t) > 0$, $u^{[1]}(t) < 0$, $u^{[2]}(t) > 0$ for $t \geq T_x$, we claim that

$$lim_{t\to\infty} u^{[i]} = l_i = 0, i = 0, 1, 2.$$

Indeed, if $l_1 < 0$, then $u'(t) \leq l_1 r(t)$ for large t,

$$u(t) \leq u(T_x) + l_1 \int_{T_x}^{t} r(t)\mathrm{d}t$$

Letting $t \to \infty$, we get a contradiction with the $u(t) > 0$. Therefore $l_1 = 0$. If $l_2 > 0$, then $(u^{[1]}(t))' \geq l_2 p(t)$ for large t

$$u^{[1]}(t) \geq u(t_0) + l_2 \int_{t_0}^{t} p(t)\mathrm{d}t$$

Letting $t \to \infty$, we get a contradiction with the $u^{[1]}(t) < 0$. Therefore $l_2 = 0$. Assume by contradiction that $l_0 > 0$, then for any $\epsilon > 0$ we have $l_0 + \epsilon > u(\mu(t)) > l_0$ for large t and choose $0 < \epsilon < \frac{l_0(1-a_0)}{a_0}$.

$$x(t) = u(t) - a(t)x(\mu(t)) > l_0 - a_0 u(\mu(t)) > l_0 - a_0(l_0 + \epsilon) = k(l_0 + \epsilon) > kl_0 \tag{2.3}$$

Where $k = \frac{l_0 - a_0(l_0+\epsilon)}{l_0+\epsilon} > 0$. In view of the fact $f(v)$ is increasing, there exists $B > 0$ such that $f(x(\delta(t))) \geq B$ for large t, hence frome equation (E) it follows that $(u^{[2]}(t))' \leq -q(t)B$. Integrating this inequality two times from t to ∞ we obtain

$$-u^{[1]}(t) \geq B\int_t^{\infty} p(v)\int_v^{\infty} q(s)\mathrm{d}s\mathrm{d}v$$

Integrating from t_1 to t we obtain

$$-u(t) + u(t_1) \geq B\int_{t_1}^{t} r(w)\int_w^{\infty} p(v)\int_v^{\infty} q(s)\mathrm{d}s\mathrm{d}v\mathrm{d}w$$

Letting $t \to \infty$ we obtain

$$\int_{t_1}^{\infty} r(w)\int_w^{\infty} p(v)\int_v^{\infty} q(s)\mathrm{d}s\mathrm{d}v\mathrm{d}w < \infty$$

We get the contradiction with condition (2.1). Therefore $l_0 = 0$ and the inequality $0 \leq x(t) \leq u(t)$ implies that $lim_{t\to\infty} x(t) = 0$.

Assume that $u(t)$ has the proper **(ii)**. Then there exists $t_1 \geq t_0$ such that $u(t) > 0$, $u^{[1]}(t) > 0$ and $u^{[2]}(t) > 0$ for $t \geq t_1$, let t_2 be such that $\delta(t) \geq t_1$ for $t \geq t_2$. Because $(u^{[2]}(t)' = -q(t)f(x(\delta(t))) < 0$ for $t \geq t_2$, $u^{[2]}(t)$ is a positive decreasing function. Integrating the equation (E) from t to ∞ we obtain

$$u^{[2]}(t) = u^{[2]}(\infty) + \int_t^{\infty} q(s)f(x(\delta(s))\mathrm{d}s$$

$$u^{[2]}(t) \geq \int_t^{\infty} q(s)f(x(\delta(s)))\mathrm{d}s \geq \lambda\int_t^{\infty} q(s)x(\delta(s))\mathrm{d}s$$

Using the (1.1) we obtain

$$u^{[2]}(t) \geq \lambda(1-a_0)\int_t^{\infty} q(s)u(\delta(s))\mathrm{d}s \geq \lambda(1-a_0)\int_t^{g(t)} q(s)u(\delta(s)))\mathrm{d}s \tag{2.4}$$

Integrating $u^{[2]}(t) = u^{[2]}(t)$ twice from t_1 to t we obtain

$$u(t) \geq \int_{t_1}^{t} r(s)\int_{t_1}^{s} p(v)u^{[2]}(v)\mathrm{d}v\mathrm{d}s$$

for $t \geq t_1$, we have

$$u(\delta(t)) \geq \int_{t_1}^{\delta(t)} r(s)\int_{t_1}^{s} p(v)u^{[2]}(v)\mathrm{d}v\mathrm{d}s$$

Substituting into (2.4) we get

$$u^{[2]}(t) \geq \lambda(1-a_0)\int_t^{g(t)} q(s)\int_{t_1}^{\delta(s)} r(v)\int_{t_1}^{v} p(w)u^{[2]}(w)\mathrm{d}w\mathrm{d}v\mathrm{d}s$$

Considering the fact that $u^{[2]}(t)$ is decreasing and $u^{[2]}(\delta(g(t)))$ is nonincreasing, we get

$$u^{[2]}(t) \geq \lambda(1-a_0)u^{[2]}(\delta(g(t))) \int_t^{g(t)} q(s) \int_{t_1}^{\delta(s)} r(v) \int_{t_1}^{v} p(w)\mathrm{d}w\mathrm{d}v\mathrm{d}s$$

Since $u^{[2]}(t)$ is decreasing. Lemma 1 holds, we have

$$1 \geq \frac{u^{[2]}(t)}{u^{[2]}(\delta(g(t)))} \geq \lambda(1-a_0) \int_t^{g(t)} q(s) \int_{t_1}^{\delta(s)} r(v) \int_{t_1}^{v} p(w)\mathrm{d}w\mathrm{d}v\mathrm{d}s$$

Which is contradiction of condition (2.2). The proof is completed.

Theorem 2. Assume that (2.1) and

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{\delta(t)} r(s)\mathrm{d}s\mathrm{d}t = \infty \tag{2.5}$$

Then any proper solution x of (E) is either oscillatory or satisfies $lim_{t\to\infty}x(t) = 0$.

Proof. By the first part of the proof of Throrem 1 any solution x tends to zero that if $u(t) = x(t) + a(t)x(\mu(t))$ has the proper **(i)**.

Without loss of generality we may assume that x is an eventually positive solution, assume that $u(t)$ has the proper **(ii)**. Then there exists $T \geq t_0$ such that $u(t) > 0$, $u^{[1]}(t) > 0$, $u^{[2]}(t) > 0$ for $t \geq T$. Since $u^{[1]}(t)$ is an eventually positive increasing function, we have $u^{[1]}(t) > u^{[1]}(T)$ and by integrating from T to t we get

$$u(t) > u^{[1]}(T) \int_T^t r(s)\mathrm{d}s = L \int_T^t r(s)\mathrm{d}s \tag{2.6}$$

Using (1.1) together with (2.6) we get

$$x(\delta(t)) \geq u(\delta(t))(1-a_0) \geq (1-a_0)L \int_T^{\delta(t)} r(s)\mathrm{d}s \tag{2.7}$$

Let $T_1 > T$ be such that $\delta(t) \geq T_1$. Integrating the equation (E) from T_1 to ∞ we obtain

$$u^{[2]}(T_1) - u^{[2]}(\infty) = \int_{T_1}^{\infty} q(s)f(x(\delta(s)))\mathrm{d}s$$

Therefore $\int_{T_1}^{\infty} q(s)f(x(\delta(s)))\mathrm{d}s < \infty$. Since (C4) holds, we have

$$\lambda \int_{T_1}^{\infty} q(s)x(\delta(s))\mathrm{d}s \leq \int_{T_1}^{\infty} q(s)f(x(\delta(s)))\mathrm{d}s$$

i.e.

$$\lambda \int_{T_1}^{\infty} q(s)x(\delta(s))\mathrm{d}s < \infty$$

and using (2.7) we get

$$\lambda(1-a_0)L\int_{T_1}^{\infty} q(t)\int_{T}^{\delta(t)} r(s)\mathrm{d}s\mathrm{d}t<\infty$$

Which contradicts (2.5). This completes the proof.

Theorem 3. Assume that $\delta(t)\le t$, $f(uv)\ge f(u)f(v)$ for $u,v\in R$, (2.1) and

$$\int_0^1 \frac{1}{f(v)}\mathrm{d}v<\infty$$

If

$$\int_{t_0}^{\infty} q(t)\int_{t_0}^{\delta(t)} r(s)\int_{t_0}^{s} p(v)\mathrm{d}v\mathrm{d}s\mathrm{d}t=\infty \tag{2.8}$$

Then any proper solution x of (E) is either oscillatory or satisfies $lim_{t\to\infty}x(t)=0$.

Proof. By the first part of the proof of Throrem 1 any solution x tends to zero that if $u(t)=x(t)+a(t)x(\mu(t))$ has the proper **(i)**.

Without loss of generality we may assume that x is an eventually positive solution, assume that $u(t)$ has the proper **(ii)**. Then there exists $T\ge t_0$ such that $u(t)>0$, $u^{[1]}(t)>0$, $u^{[2]}(t)>0$ for all $t\ge T$. Because of $u^{[2]}$ is decreasing, we get

$$u^{[1]}(t)=u^{[1]}(t_1)+\int_{t_1}^{t} p(s)u^{[2]}(s)\mathrm{d}s\ge u^{[2]}(t)\int_{t_1}^{t} p(s)\mathrm{d}s$$

and therefore

$$u'(t)\ge u^{[2]}(t)r(t)\int_{t_1}^{t} p(s)\mathrm{d}s$$

$$u(t)\ge u(t)-u(t_1)=\int_{t_1}^{t} u'(s)\mathrm{d}s\ge u^{[2]}(t)\int_{t_1}^{t} r(s)\int_{t_1}^{s} p(v)\mathrm{d}v\mathrm{d}s \tag{2.9}$$

Using (E) and (1.1) we get

$$-(u^{[2]}(t))'=q(t)f(x(\delta(t)))\ge q(t)f(1-a_0)f(u(\delta(t)))$$

Using (C4) and ((2.9) we get

$$-(u^{[2]}(t))'\ge \lambda q(t)f(1-a_0)f(u^{[2]}(t))\int_{t_1}^{\delta(t)} r(s)\int_{t_1}^{s} p(v)\mathrm{d}v\mathrm{d}s$$

Hence

$$-\int_{t_1}^{t}\frac{u^{[2]}(t))'}{f(u^{[2]}(t))}\mathrm{d}t\ge \lambda f(1-a_0)\int_{t_1}^{t} q(w)\int_{t_1}^{\delta(w)} r(s)\int_{t_1}^{s} p(v)\mathrm{d}v\mathrm{d}s\mathrm{d}w$$

Letting $t\to\infty$

$$-\int_{t_1}^{\infty}\frac{u^{[2]}(t))'}{f(u^{[2]}(t))}\mathrm{d}t=\int_{u^{[2]}(\infty)}^{u^{[2]}(t_1)}\frac{ds}{f(s)}<\infty$$

We get the contradiction with condition (2.8). The proof is completed.

3 Examples

Example 1. Consider the equation

$$\left(\frac{1}{t}[x(t)+\frac{1}{3t}x(\frac{t}{2})]'\right)''+\frac{1}{t^3}x(k^2t)=0, \quad t\geq 1 \tag{3.1}$$

where $0<k<1$. If we take $g(t)=\frac{t}{k}$. One can check that condition (2.1) and (2.2) are satisfied. Thus, by Theorem 1, then any proper solution x of (3.1) is either oscillatory or satisfies $lim_{t\to\infty}x(t)=0$.

Example 2. Consider the equation

$$\left(\frac{1}{t}[x(t)+\frac{1}{5t}x(\frac{t}{2})]'\right)''+\frac{1}{t^3}x(\frac{t}{3})=0, \quad t\geq 1 \tag{3.2}$$

One can check that condition (2.1) and (2.5) are satisfied. Thus, by Theorem 2, then any proper solution x of (3.2) is either oscillatory or satisfies $lim_{t\to\infty}x(t)=0$.

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