

# Accurate time calculations of falling bodies in the Earth's gravitational field and comparisons with Newton's laws of vertical motion

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## Abstract

The Earth is exposed annually to the fall of some meteorites and probably other celestial bodies which cause a potential danger to vital areas in several countries. Consequently, the accurate calculation of the falling time of such bodies is useful in order to take the necessary procedures for protecting these areas. In this paper, Newton's law of general gravitation is applied to analyze the vertical motion in the Earth's gravitational field. The falling time is obtained in exact form. The results are applied on several objects in real life.

*Keywords:* Newton's laws; vertical motion; general gravitation; falling time.

## 1 Introduction

Several centuries ago, the British scientist Isaac Newton developed the laws of regular motion with a constant acceleration in a straight line. Such laws are often studied in the early years of the university stage to investigate the vertical motion of objects close enough to the surface of the earth, that is, at small heights compared to the radius of the Earth. Newton also discovered his important law of general gravitation in classical mechanics, which is usually used to analyze the motion of an object in the gravitational

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field of another object. The latter is of course more general than the aforementioned vertical motion laws. The question that we want to answer in the present study is that; what is the difference between the falling time of an object in view of both Newton's laws of vertical motion and Newton's law of general gravitation? In the present study, we will determine the amount of error resulting from the applications of Newton's laws of vertical motion. Such error will be expressed in terms of the height from which the object fall. The results will be applied on several objects in real life.

Perhaps the first attempt to study the falling objects in the Earth's gravitational field was the experiment of the pound and the quill made by the great scientist Isaac Newton several centuries ago. Newton concluded from his experiment that the falling time of two bodies from the same height does not depend on their masses and that they will take the same time to reach the ground in the absence of air resistance, where Newton has conducted his famous experiment in a vacuum tube of air. Newton made great contributions and discovered many scientific laws in classical mechanics [1], not only, he also developed other important theories and laws in various branches of physics and astronomy. Some of the most famous laws developed by Newton were the three laws of motion in a straight line with a constant/regular acceleration. These laws are usually taught in the early years for the students in physics and mathematics departments. Replacing the constant acceleration with the acceleration due to the gravity of Earth leads to Newton's laws of vertical motion which can be used to study the vertical motion of objects near to the Earth's surface.

The question arises here is that; is it possible to apply Newton's laws of vertical motion on objects falling from hundreds of kilometers above the ground? On the other hand, Newton derived his famous law of gravitation through which can be used to study the vertical motion of objects far away from the Earth's surface [2]. Hence, Newton's law of gravitation could be applied in a much greater range than Newton's laws of vertical motion. The questions that we try to answer in the current study are; what is the difference between the results that can be obtained in light of both Newton's laws of vertical motion and Newton's law of gravitation? Is the falling time derived from both types of laws will be the same when an object falls from a prescribed height above the ground? In the present research, we will be able to determine the amount of error resulting from the applications of Newton's laws of vertical motion in estimating the falling time of objects as a function of the height.

## 2 Formulation of the problem

The equation of motion of a particle in a general resistant medium in view of Newton's law of gravitation is given by [2]

$$\ddot{r}(t) = -\frac{GM}{r^2} + kv^n, \quad (1)$$

where,  $M$  is the mass of Earth,  $G$  is Newton's constant of general gravitation,  $r$  is the distance of the particle from the center of Earth,  $v(t) = \dot{r}(t)$  is the vertical instantaneous velocity of the particle,  $k$  is the constant of resistance, and  $n$  is a positive natural number. Assuming that  $R$  is the radius of Earth, the initial conditions (ICs) are given as

$$\dot{r}(0) = 0, \quad r(0) = h + R, \quad (2)$$

where  $h$  is the height of the particle above the Earth's surface at initial time. The proposed method depends basically on applying some basic concepts in calculus [3,4] for the special case  $k = 0$ . In case  $k \neq 0, n \geq 1$ , the Adomian decomposition method (ADM) [5-14] may be applied to solve the nonlinear system (1-2) which is a complex nonlinear initial value problem. The objectives of this paper are focused on estimating the falling time and comparing our results with the corresponding ones obtained from Newton's laws of vertical motion. Then, applying the results on several bodies in our real life.

## 3 The exact solution

In this section, the exact solution of the system (1-2) will be obtained at the special case  $k = 0$ . In this case, Eq. (1) becomes

$$\ddot{r}(t) = -\frac{GM}{r^2}. \quad (3)$$

Multiplying both sides by  $\dot{r}$ , we have

$$\dot{r}\ddot{r}(t) = -GMr^{-2}\dot{r}. \quad (4)$$

Integrating once, yields

$$\frac{1}{2}(\dot{r}(t))^2 = \frac{GM}{r} + c_1, \quad (5)$$

where  $c_1$  is a constant of integration. Applying the ICs (2), we obtain

$$c_1 = -\frac{GM}{h + R}, \quad (6)$$

and Eq. (5) becomes

$$(\dot{r}(t))^2 = 2GM \left( \frac{1}{r} - \frac{1}{h+R} \right), \quad (7)$$

or

$$\dot{r}(t) = \pm \sqrt{2GM \left( \frac{1}{r} - \frac{1}{h+R} \right)} \quad (8)$$

Since  $r(t)$  is a decreasing function in time, we choose the negative sign in (8) and this yields

$$\frac{dr}{dt} = -\sqrt{2GM \left( \frac{1}{r} - \frac{1}{h+R} \right)}. \quad (9)$$

Making use of the new variable  $u = \frac{1}{r}$ , we have

$$\frac{dr}{dt} = -\frac{1}{u^2} \frac{du}{dt}. \quad (10)$$

Inserting (10) into (9) gives

$$-\frac{1}{u^2} \frac{du}{dt} = -\sqrt{2GM(u - \alpha)}, \quad (11)$$

where

$$\alpha = \frac{1}{h+R}. \quad (12)$$

Using separation of variables approach, we can write (11) as

$$\frac{du}{u^2 \sqrt{u - \alpha}} = \sqrt{2GM} dt. \quad (13)$$

Integrating once again, yields

$$\int \frac{du}{u^2 \sqrt{u - \alpha}} = \sqrt{2GM} t + c_2, \quad (14)$$

where  $c_2$  is also a constant of integration. Implementing the trigonometric substitution method, we have

$$u = \alpha (\sec \phi)^2, \quad du = 2\alpha (\sec \phi)^2 \tan \phi d\phi. \quad (15)$$

Substituting (15) into (14) and simplifying leads to

$$\frac{2}{\sqrt{\alpha^3}} \int (\cos \phi)^2 d\phi = \sqrt{2GM} t + c_2, \quad (16)$$

i.e.,

$$\frac{1}{\sqrt{\alpha^3}} (\phi + \sin \phi \cos \phi) = \sqrt{2GM} t + c_2. \quad (17)$$

From (15), we have

$$\phi = \sec^{-1} \left( \sqrt{\frac{u}{\alpha}} \right), \quad \cos \phi = \sqrt{\frac{\alpha}{u}}, \quad \sin \phi = \sqrt{1 - \frac{\alpha}{u}}. \quad (18)$$

Hence,

$$\frac{1}{\sqrt{\alpha^3}} \left[ \sec^{-1} \left( \sqrt{\frac{u}{\alpha}} \right) + \sqrt{\frac{\alpha}{u}} \sqrt{1 - \frac{\alpha}{u}} \right] = \sqrt{2GM} t + c_2, \quad (19)$$

or

$$\sqrt{(h+R)^3} \left[ \sec^{-1} \left( \sqrt{\frac{u}{\alpha}} \right) + \sqrt{\frac{\alpha}{u}} \sqrt{1 - \frac{\alpha}{u}} \right] = \sqrt{2GM} t + c_2, \quad (20)$$

From the ICs (2), we have

$$u(0) = \frac{1}{r(0)} = \frac{1}{h+R} = \alpha, \quad (21)$$

Applying this condition on Eq. (20), we obtain

$$c_2 = \sqrt{(h+R)^3} \sec^{-1}(1) = 0, \quad (22)$$

and hence,

$$\sqrt{(h+R)^3} \left[ \sec^{-1} \left( \sqrt{\frac{u}{\alpha}} \right) + \sqrt{\frac{\alpha}{u}} \sqrt{1 - \frac{\alpha}{u}} \right] = \sqrt{2GM} t. \quad (23)$$

The product  $GM$  is also given by

$$GM = gR^2, \quad (24)$$

where  $R$  is the radius of the Earth and  $g$  is the acceleration due to gravity of the Earth. Accordingly, the equation (22) gives the falling time as

$$t = \sqrt{\frac{(h+R)^3}{2gR^2}} \left[ \sec^{-1} \left( \sqrt{\frac{u}{\alpha}} \right) + \sqrt{\frac{\alpha}{u}} \sqrt{1 - \frac{\alpha}{u}} \right], \quad (25)$$

or in terms of  $r$  and  $h$  as

$$t = \sqrt{\frac{(h+R)^3}{2gR^2}} \left[ \sec^{-1} \left( \sqrt{\frac{h+R}{r}} \right) + \sqrt{\frac{r}{h+R}} \sqrt{1 - \frac{r}{h+R}} \right]. \quad (26)$$

The object reaches the Earth's surface when  $r = R$  and accordingly the exact falling time  $T_{\text{Exact}}$  is expressed as

$$T_{\text{Exact}} = \sqrt{\frac{(h+R)^3}{2gR^2}} \left[ \sec^{-1} \left( \sqrt{\frac{h+R}{R}} \right) + \sqrt{\frac{R}{h+R}} \sqrt{1 - \frac{R}{h+R}} \right]. \quad (27)$$

Newton's laws of vertical motion for a falling object are well-known as

$$v = v_0 + gt, \quad v_0 = \dot{r}(0) = 0, \quad (28)$$

$$h = v_0 t + \frac{1}{2}gt^2, \quad (29)$$

$$v^2 = v_0^2 + 2gh. \quad (30)$$

From Eq. (30), the approximate falling time  $T_{\text{Approx}}$  is expressed as

$$T_{\text{Approx}} = \sqrt{\frac{2h}{g}}. \quad (31)$$

Therefore, the error  $\text{Error}(t)$  in estimating the falling time is given by

$$\text{Error}(t) = T_{\text{Exact}} - T_{\text{Approx}}, \quad (32)$$

and in terms of the height  $h$  we have

$$\text{Error}(t) = \sqrt{\frac{(h+R)^3}{2gR^2}} \left[ \sec^{-1} \left( \sqrt{\frac{h+R}{R}} \right) + \sqrt{\frac{R}{h+R}} \sqrt{1 - \frac{R}{h+R}} \right] - \sqrt{\frac{2h}{g}}. \quad (33)$$

## 4 Applications

In this section, we give some applications of the exact formula of the falling time for various objects in real life. The following values of the radius of Earth and the acceleration due to gravity of Earth are implemented [15] to conduct the results of this section:

$$\begin{aligned} R|_{\text{at the equator}} &= 6378.137 \times 10^3 \text{ [meter]}, & R|_{\text{at the poles}} &= 6356.752 \times 10^3 \text{ [meter]}, \\ g &= 9.7803253359 \left( \frac{1 + 0.001931850400 \sin^2 \lambda}{\sqrt{1 - 0.006694384442 \sin^2 \lambda}} \right) \text{ [meter/s}^2\text{]}, \end{aligned} \quad (34)$$

where  $\lambda$  denotes the latitude and  $\lambda \in [0, \pi]$ . At the equator, we have  $g \approx 9.78 \text{ [meter/s}^2\text{]}$  ( $\lambda = 0$ ) while at the poles of Earth we find that  $g \approx 9.83 \text{ [meter/s}^2\text{]}$  ( $\lambda = \pi/2$ ). The variation of the acceleration due to gravity versus the latitude  $\lambda$  is depicted in Fig. 1.

### 4.1 Aeroplane

It is well known in the fields of aviation and air transport that the aircrafts, which are used in internal or international flights, fly at altitudes ranging between 29000 and 35000

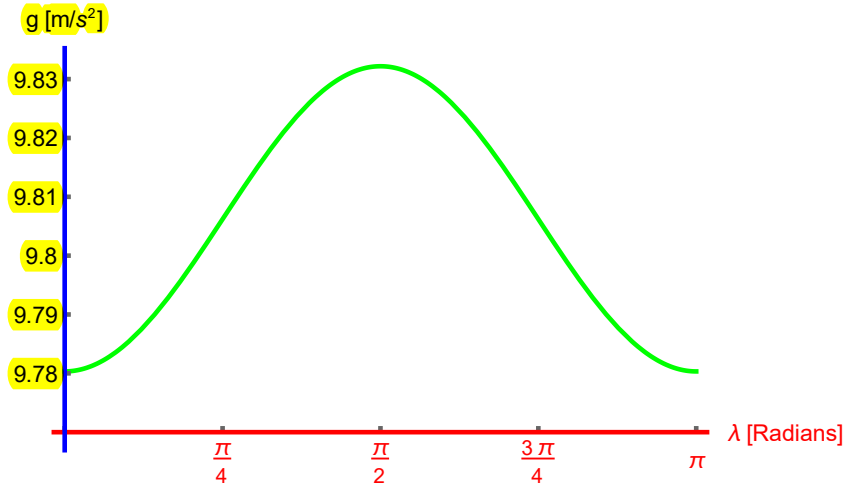


Figure 1: Variation of the acceleration due to gravity of Earth  $g$  versus the latitude  $\lambda$ .

feet, equivalent to 9 to 11 kilometers above the Earth's surface. Perhaps the reason for flying at this altitude is that such a layer of the Earth's atmosphere is more stable than other layers and air resistance is lower the required thrust is the lowest and the same for fuel consumption. We are now facing the question that if all of the plane's engines are suddenly stopped, how long will the plane take to reach the ground? To answer that question, let us consider that the average altitude of aircraft is 10 kilometers above the Earth's surface, and by substituting for  $h = 10$  [Km] or  $h = 10000$  [m] in Eq. (27) we obtain  $T_{\text{Exact}} = 45.2806 \approx 45$  seconds, which means that the aircraft takes about 45 seconds to reach the ground in the absence of air resistance. Such result coincides with the time when the atomic bomb fell on the cities of Hiroshima and Nagasaki in Japan during the second world war. Applying the approximate formula (31), we have  $T_{\text{Approx}} = 45.2216 \approx 45$  seconds. The error (32) in this case is too small and given as  $\text{Error} = T_{\text{Exact}} - T_{\text{Approx}} = 45.2806 - 45.2216 = 0.0590$  seconds This is because the height  $h = 10$  [Km] is too small when compared with the radius of the Earth.

## 4.2 Geostationary Satellites

The geostationary satellites are at altitude  $h = 36000$  [Km] above the Earth's surface and they are in stationary orbits around the Earth. Assume that the motion of such satellites is suddenly stopped, regardless of how this happens, then the expected exact time taken by these satellites to reach the Earth's surface is calculated from (27) as  $T_{\text{Exact}} = 14962.8$  seconds. Converting this value into hours and minutes gives  $T_{\text{Exact}} = 4$  hours and 9 minutes.

### 4.3 The Moon around Earth

The Moon is at a distance 384400 Km from the center of the Earth, consequently, the corresponding height is  $h = 384400 - 6378.137 = 378021.863$  [Km]. Substituting  $h = 378021.863$  [Km] in (27), we get  $T_{\text{Exact}} = 419295.86$  seconds. Converting this value into days and hours gives  $T_{\text{Exact}} = 4$  days and 20 hours. This result gives the falling time of the Moon on the Earth (assuming that the Moon is suddenly stopped, whatever the reason) which coincides with Ref. [16] (Problem 5.107, page 141).

### 4.4 The Earth around Sun

The Earth is at a distance  $150 \times 10^6$  Km from the center of the Sun. In order to estimate the falling time of the Earth on the Sun (assuming that the Earth is suddenly stopped, whatever the reason), we modify Eq. (27) as

$$T_{\text{Exact}} = \sqrt{\frac{(h + R_s)^3}{2g_s R_s^2}} \left[ \sec^{-1} \left( \sqrt{\frac{h + R_s}{R_s}} \right) + \sqrt{\frac{R_s}{h + R_s}} \sqrt{1 - \frac{R_s}{h + R_s}} \right], \quad (35)$$

where  $R_s$  is the radius of the Sun ( $R_s = 6.96 \times 10^5$  Km) and  $g_s$  is the acceleration of gravity due to the Sun ( $g_s = 273$  [meter/s<sup>2</sup>]). In this case, the height of Earth above the Sun equals  $h = 150 \times 10^6 - 6.96 \times 10^5 = 149.304 \times 10^6$  [Km]. Accordingly, Eq. (35) leads to  $T_{\text{Exact}} = 5.61039 \times 10^6$  seconds. Converting this value into days gives  $T_{\text{Exact}} = 64.9 \approx 65$  days. Also, this result agrees with the obtained result in Ref. [16] (Problem 5.108, page 141).

## 5 Discussion of errors

Using the error equation (33), we present in Tables (1-3) some numerical results, from which it becomes clear that the amount of error in time is about 21 seconds in the first 500 kilometers above the surface of the earth, as in Table 1.

Table 1: Calculated errors for  $h = 100, 200, 300, 400, 500$  Km.

$h$ [Km]	100	200	300	400	500
Error( $h$ )[S]	1.86753	5.27977	9.69521	14.9202	20.8426

While the error in time is about 59 seconds (approximately one minute), in the first 1000 km, as shown in Table 2.

Table 2: Calculated errors for  $h = 600, 700, 800, 900, 1000$  Km.



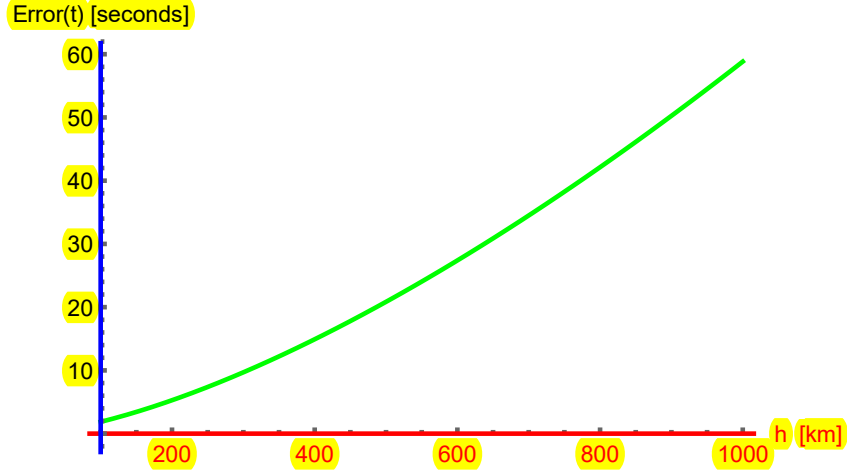


Figure 2: Graph of the estimated error in the falling time, given by Eq. (33), versus the height  $h$  in the range  $h \in [100, 1000]$  [km].

$h$ [Km]	600	700	800	900	1000
Error( $h$ )[S]	27.3867	34.4968	42.1298	50.2509	58.8314

It is clear from Table 3 that the amount of error in time is approximately 853 seconds, that is, 14 minutes if the altitude reaches six thousand kilometers.

Table 3: Calculated errors for  $h = 2000, 3000, 4000, 5000, 6000$  Km.

$h$ [Km]	2000	3000	4000	5000	6000
Error( $h$ )[S]	165.802	303.663	466.272	650.11	852.817

In addition, it can be indicated from Eq. (33) that the error in time is approximately 12249.5 seconds, i.e. about three hours and 24 minutes if the body fell from a height of 36,000 km above the surface of the Earth, which is the same height as the motion of geosynchronous satellites. In light of these results, it becomes clear to us that it is preferable not to apply Newton's laws of vertical motion at altitudes higher than thousand kilometer above the surface of the Earth. This is because the amount of error (in the estimated time of falling objects) becomes minutes and increases to hours with increasing the height above the surface of the Earth. This conclusion can be confirmed through Fig. 2, Fig. 3, and Fig. 4.

## 6 Conclusion

In this paper, Newton's law of general gravitation was applied to analyze the vertical motion of an object towards the Earth. Explicit formula for the falling time was obtained

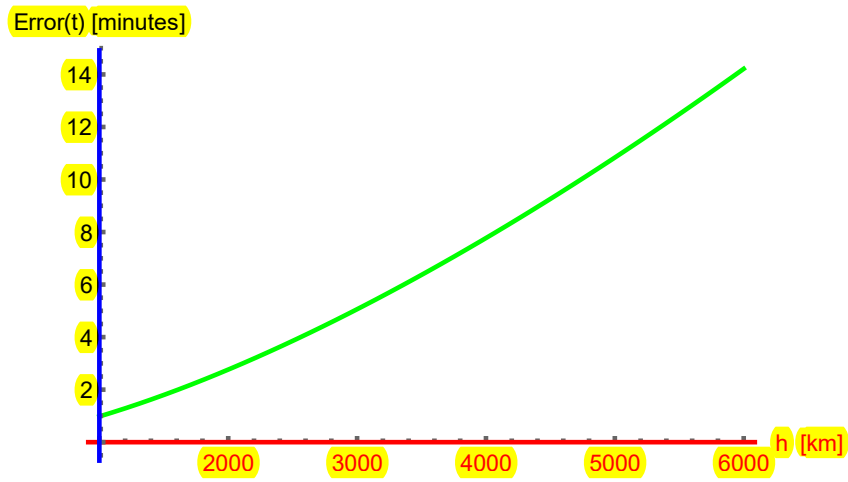


Figure 3: Graph of the estimated error in the falling time, given by Eq. (33), versus the height  $h$  in the range  $h \in [1000, 6000]$  [km].

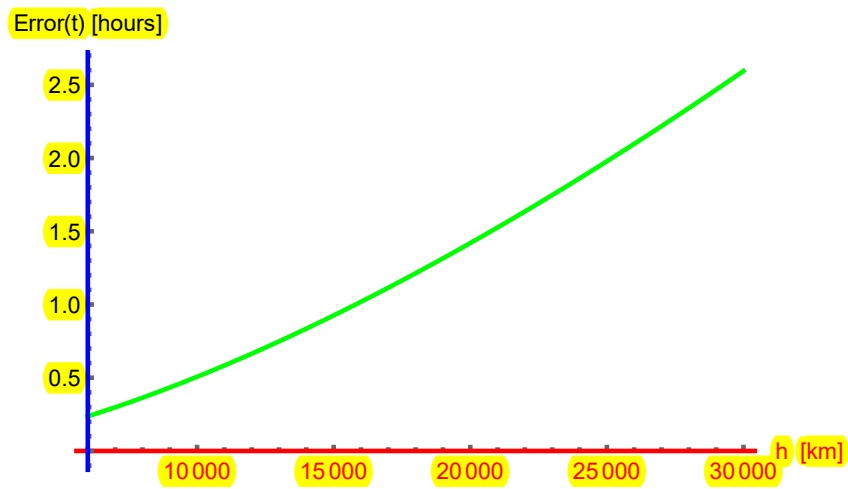


Figure 4: Graph of the estimated error in the falling time, given by Eq. (33), versus the height  $h$  in the range  $h \in [6000, 30000]$  [km].

and applied on some objects in real life. The results revealed that the time taken by a plane to reach the ground was about 45 seconds in the absence of air resistance (if all of the plane's engines are suddenly stopped). The geostationary satellites (altitude  $h = 36000$  [Km] above the Earth's surface) reach the Earth's surface in 4 hours and 9 minutes under the assumption that the motion of such satellites is suddenly stopped and regardless of how this happens. Under such assumption, the Moon takes about 4 days and 20 hours to fall on the Earth. Furthermore, the time taken by the Earth to reach the Sun's surface was about 65 days. The last two results were in full agreement with the calculations made in Ref. [16] (Problems 5.107 and 5.108, page 141). Finally, the error resulting from applying Newton's laws of vertical motion was obtained in terms of the height. In view of the obtained results, it was recommended to avoid the use of Newton's laws of vertical motion at altitudes higher than thousand kilometer above the Earth. This is because the error in the estimated falling time becomes minutes and increases to hours if the height is greater than thousand kilometer.

**Conflicts of Interest:** The authors declare no conflict of interest.

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## References

- [1] Anthony Bedford, Wallace Fowler, Engineering Mechanics, Six Edition, Addison Wesley Longman, 1996.
- [2] A.N.Matveev, Mechanics and Theory of Relativity, Mir Publishers, Moscow, 1989.
- [3] Klaus Weltner, Wolfgang J. Weber, Jean Grosjean, Peter Schuster, Mathematics for Physicists and Engineers, Springer, 2009.
- [4] S. Selcuk Bayin, Mathematical Methods for Science and Engineering, Wiely, 2006.
- [5] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Acad, Boston, 1994.
- [6] A.M. Wazwaz, Adomian decomposition method for a reliable treatment of the Bratu type equations, Appl. Math. Comput., 166 (2005), 652-663.

- [7] A. Ebaid, Approximate analytical solution of a nonlinear boundary value problem and its application in fluid mechanics, *Z. Naturforschung A.*, 66 (2011), 423-426.
- [8] A. Ebaid, A new analytical and numerical treatment for singular two-point boundary value problems via the Adomian decomposition method, *J. Comput. Appl. Math.*, 235 (2011), 1914-1924.
- [9] E.H. Ali, A. Ebaid, R. Rach, Advances in the Adomian decomposition method for solving two-point nonlinear boundary value problems with Neumann boundary conditions, *Comput. Math. Applic.*, 63 (2012), 1056-1065.
- [10] A. Ebaid, M.D. Aljoufi, A.-M. Wazwaz, An advanced study on the solution of nanofluid flow problems via Adomian's method, *Appl. Math. Lett.*, 46 (2015), 117-122.
- [11] A. Alshaery, A. Ebaid, Accurate analytical periodic solution of the elliptical Kepler equation using the Adomian decomposition method, *Acta Astronautica*, 140 (2017), 27-33.
- [12] H.O. Bakodah, A. Ebaid, Exact Solution of Ambartsumian Delay Differential Equation and Comparison with Daftardar-Gejji and Jafari Approximate Method, *Mathematics*, 2018, 6(12), 331; <https://doi.org/10.3390/math6120331>.
- [13] H.O. Bakodah, M.A. Banaja, B.A. Alrigi, A. Ebaid, R. Rach, An efficient modification of the decomposition method with a convergence parameter for solving Korteweg de Vries equations, *Journal of King Saud University-Science*, 31(4) (2019), 1424-1430.
- [14] Li, W., Pang, Y. Application of Adomian decomposition method to nonlinear systems. *Adv Differ Equ* 2020, 67 (2020). <https://doi.org/10.1186/s13662-020-2529-y>.
- [15] Department of Defense World Geodetic System 1984 — Its Definition and Relationships with Local Geodetic Systems, NIMA TR8350.2, 3rd ed., Tbl. 3.4, Eq. 4-1.
- [16] Murray R. Spiegel, Schaum's outline series of theoretical Mechanics, McGRAW-HILL BOOK COMPANY, New York, 1967.