On Generalized Moment Exponential distribution and Power Series Distribution

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ABSTRACT

In this research paper, a new life time family is introduced. Sadaf (2014) proposed a moment exponential power series (MEPS) distribution. Generalized moment exponential power series (GMEPS) distribution is a general form of MEPS distribution. It is characterized by compounding GME distribution and power series (PS) distribution. This new family has some new sub models such as GME geometric distribution, GME Poisson (GMEP) distribution, GME logarithmic (GMEL) distribution and GME binomial (GMEB) distribution. We provide statistical properties of GMEPS family of distributions. We find here expression of quantile function based on Lambert W function, the density function of rth order statistic and moments of GMEPS distribution. Descriptive expressions of Shannon entropy and Rényi entropy of new general model are found. We provide special sub-models of the GMEPS family of distributions. The maximum likelihood (ML) estimation method is used to find estimates of the parameters of GMEPS distribution. Simulation study is carried out to check the convergence of new estimators. We apply GMEPS family of distributions on two sets of real data.

KEYWORDS

Compound family; Moment exponential distribution; GME distribution; lifetime distribution; PS distribution; order statistics.

1. INTRODUCTION

In literature, probability distributions to model lifetime data are based on two parameters' PS probability distributions, e.g. Marshal and Olkin (1997) introduced two parameter exponential distribution and proved it a competitor of Weibull, gamma and log-normal distribution. Adamidis and Loukas (1998) introduced exponential geometric (EG) and applied it on real lifetime data. The other two parameter lifetime distributions are exponential Poisson (EP) distribution (Kus, 2007) and exponential logarithmic (EL) distribution (Tahmasbi and Rezaei (2008)). Chahkandi and Ganjali (2009) introduced the exponential PS family of distributions. Exponential PS family contain the EG distribution, EP distribution and EL distribution as sub-models. Adamidis et al. (2005) introduced the extended EG distribution. The Extended EG distribution has the constant, decreasing and increasing hazard function. Barreto-Souza et al. (2010) presented the Weibull geometric (WG) distribution. They discussed several properties of this model. It is the extended form of EG distribution. Barreto-Souza and Cribari-Neto (2009) and Silva et al. (2013) developed the general forms of the EP distribution and EG distribution respectively. Lemos- Morais and Barreto-Souza (2011) introduced the Weibull PS family of distributions. The Weibull PS family of distributions contain Weibull geometric, Weibull Poisson distribution and Weibull logarithmic distribution.

Mahmoudi and Jafari (2012) developed the generalized exponential PS distribution which is an extended form of exponential form of exponential PS distribution. Sandhya and Prasanth (2014) introduced Marshall-Olkin discrete uniform distribution and discussed its theoretical properties.

This model is also an alternative of Weibull PS distribution. Silva and Cordeiro (2015) introduced Burr XII PS distribution and applied it to real life data. Dara (2012) introduced ME distribution as:

$$g(y;\beta)=\beta^2 y e^{-\beta y}, \quad y,\beta>0. \tag{1}$$

Iqbal and Ali (2017) applied the transformation $Y=X^{\alpha}$, in (1) and developed GME distribution

$$g(x;\alpha,\beta)=\alpha\beta^2 x^{2\alpha-1}e^{-\beta x^{\alpha}}, \; x,\alpha,\beta>0. \tag{2}$$

Noack (1950) introduced PS family of discrete distributions which contain discrete distributions like binomial distribution, geometric distribution, logarithmic distribution and Poisson distribution. Suppose Z is a discrete random variable truncated at zero, the probability mass function of Z is:

$$P(Z=z;\theta)=\frac{a_z\theta^z}{M(\theta)}, \; z=1,2,3..., \tag{3}$$

where $M(\theta)=\sum_{z=1}^{\infty}a_z\theta^z$, $z=1,2,3...,$

and θ is the scale parameter.

In this article, we introduce generalized moment exponential PS distributions. The shapes of generalized moment exponential *PS* family of distributions are bathtub, increasing, decreasing and constant for various values of parameters, therefore, it can use in the research areas of reliability and engineering. The new generalized moment exponential PS family of distributions has flexibility in a real data modeling. Moreover, the model *GMEG* i.e. member of *GMEPS* family of distributions showed significantly better in fitting on lifetime data than Weibull distribution, exponential distribution and EE distribution.

The contents of this research article are arranged as follows: Section 2 deals with derivation of generalized moment exponential PS (GMEPS) distribution with some basic statistical properties and hazard function. Section 3 contains the expressions of quantile function based on Lambert W function, moments of GMEPS, Shannon entropy and Rényi entropy of new general model. Section 4 related to some special sub-models of *GMEPS* distribution. Section 5 contains maximum likelihood (ML) estimators for the unknown parameters on the basis of the family and a simulation study is carried out on the basis of ML estimates. In Section 6, *GMEG* distribution is applied on two data sets [Murthy et al.;2004, Bjerkedal ;1960] and comparison is made with existing lifetime distributions**.** Finally, Section 7 is devoted for some concluding remarks.

2. NEW FAMILY OF DISTRIBUTIONS

In this section, the *GMEPS* family of new distributions is derived. We use the compounding technique to find this new family and it is derived by compounding *GME* distribution and *PS* distributions.

Let $X_i,\ 1 \le i \le n$ be iid r.v's having *GME* distribution with *pdf* (1) and the following *cdf*:

$$G(x;\alpha,\beta) = 1-(1+\beta x^{\alpha})\ e^{-\beta x^{\alpha}}$$

$$G(x;\alpha,\beta) = 1-H(x;\alpha,\beta) \text{ where } H(x;\alpha,\beta) = (1+\beta x^{\alpha})\ e^{-\beta x^{\alpha}}$$

Suppose that Z has a zero truncated PS distribution with the probability mass function (pmf) (3). Let $X_{(1)} = \min\{X_1, X_2, ..., X_z\}$ independent of $X\,'s$, then the probability density function of $X_{(1)} | Z$ is as:

$$f_{X_{(1)}|Z}(x|\, z;\alpha,\beta) = z\; g(x;\alpha,\beta)\left(H(x;\alpha,\beta)\right)^{z-1}.$$

The following function is the joint *pdf* of $X_{(1)}$ and Z :

$$f_{X_{(1)}Z}(xz;\alpha,\beta) = \frac{z\; a_z \theta^z g(x;\alpha,\beta)\left(H(x;\alpha,\beta)\right)^{z-1}}{M(\theta)}.$$

The probability density function of a *GMEPS* family of distributions is,

$$f(x;\Theta) = \frac{\theta g(x;\alpha,\beta) M'\left(\theta H(x)\right)}{M(\theta)}, x, \alpha, \beta, \theta, > 0. \qquad (4)$$

where $\Theta \equiv (\alpha,\beta,\theta)$ is a set of parameters and $M(\theta)$ defined in (3). And a continuous random variable X with *pdf* (4) is denoted by X~$GMEPS(\alpha,\beta,\theta)$. with α and θ are shapes parameters and β is a scale parameter.

Furthermore, the *cdf* of *GMEPS* family of distributions corresponding to (4) is obtained as

$$F(x;\Theta) = \int_0^x f\left(t,\Theta\right) = 1 - \frac{M'\left(\theta H\left(x\right)\right)}{M(\theta)}. \qquad (5)$$

it can easily be proved $M'\left(\theta H\left(x\right)\right) = -\theta g\left(x;\alpha,\beta\right)$

If $\alpha = 1$, the *GMEPS* family is reduced to *MEPS* (Sadaf (2014)). Equations (6) and (7) contain the reliability of *GMEPS'* distributions and hazard rate functions for *GMEPS'* distributions respectively.

$$R(x;\Theta) = \frac{M\left(\theta H(x)\right)}{M(\theta)}, \qquad (6)$$

and,

$$h(x;\Theta)=\frac{\theta g(x;\alpha,\beta)M'\left(\theta H(x)\right)}{M\left(\theta H(x)\right)}. \qquad (7)$$

3. STATISTICAL PROPERTIES

In this section, we obtain expressions of some statistical properties of *GMEPS* family of distributions. We deduce two propositions. The first proposition indicates that *GME* distribution is the limiting form of the *GMEPS* family of distributions. And second proposition gives expansion of *GMEPS* distribution.

Proposition (1)

The *GME* distribution is a limiting case of *GMEPS'* distributions when $\theta \to 0^+$.

Proof:

By applying $P(\theta)=\sum_{z=1}^{\infty} a_z\theta^z$, for $x>0$ in cdf (4), then we obtain

$$\lim_{\theta\to 0^+} F(x;\Theta)=1-\lim_{\theta\to 0^+}\frac{P\left(\theta H(x)\right)}{P\left(\theta\right)}.$$

By using L.H. rule, we have

$$\lim_{\theta\to 0^+} F(x;\Theta)=1-\frac{H(x)[1+a_1^{-1}\lim_{\theta\to 0^+}\sum_{z=2}^{\infty} za_z\left(\theta H(x)\right)^{z-1}]}{1+a_1^{-1}\lim_{\theta\to 0^+}\sum_{z=2}^{\infty} za_z\theta^{z-1}}=G\left(x;\alpha,\beta\right)$$

which is the *cdf* of the *GME* distribution (3).

Proposition (2)

The density function of *GMEPS* family of distribution can be expressed in the PDF of lower order statistics $X_{(1)}$.

Proof

Since $f'(\theta)=\sum_{z=1}^{\infty} z a_z \theta^{z-1}$, then the pdf (3) can be expressed as follows

$$f(x;\psi)=\sum_{z=1}^{\infty} P(Z=z;\theta) g_{x_{(1)}}(x;z),$$

where $g_{x_{(1)}}(x;z)$ is the pdf of $X_{(1)}$ given by

$$g_{X_{(1)}}(x;z)=z(1+\beta x^{\alpha})^{z-1} e^{-(z-1)\beta x^{\alpha}} g\left(x;\alpha,\beta\right),\ x,\alpha,\beta>0.$$

3.2 The Lambert W function

Lambert (1758) and Euler (1779) both developed the Lambert W function. In Algebra, Lambert W function is a standard word and formula and it used to find the solution of special form of equation. Corless et al. (1996) gave almost complete survey of this function. This function is a solution of the following equation based on complex number

$$W(z)\exp\left(W(z)\right)=z$$

The W(z) has two real branches according to negative and positive intervals of Z.

Lemma 1:- Let a, b *and* c be three numbers of complex type, the equation $z+ab^{z}=c$ has the solution

$$z=c-\frac{1}{\log(b)} W\left(ab^{c}\log(b)\right)$$

where W denotes the well-known lambert W function and $z \in C$

3.2.1 Quantile function of the new GMEPS family

This subsection contains the derivation of $Q(p)$, the $Q(p)$ is known as quantile function of the *GMEPS* distribution at p. This function is defined by $Q(p) = p$, and is the root of the following equation

$$1-\frac{M\left(\theta\,\bar{G}(Q(p))\right)}{M(\theta)} = p,\ 0<p<1.\ where\ \bar{G}(x) = 1-G(x)$$

Suppose $B(p) = -(1+\beta(Q(p))^{\alpha})$, and after some simplification we have the equation

$$B(p)e^{B(p)} = -\frac{M^{-1}\left((1-p)M(\theta)\right)}{\theta e^{1}},$$ the solution of $B(p)$ is

$$B(p)e^{B(p)} = W[-\frac{M^{-1}\left((1-p)M(\theta)\right)}{\theta e^{1}}],$$

Consequently, the $Q(p)$ of the *GMEPS* family is given by solving the following equation for $Q(p)$.

$$(Q(p)) = \left(-\frac{1}{\beta} - W[-\frac{M^{-1}\left((1-p)M(\theta)\right)}{\theta e^{1}}]\right)^{1/\alpha}. \tag{8}$$

3.3 Moments and moment generating function

The rth moment of X for the *GMEPS* family of distribution, is

$$\mu_r' = \sum_{z=1}^{\infty} P(Z=z;\theta)\int_0^{\infty} x^r g_{X_{(1)}}(x;z)dx.$$

Then,

$$\mu_r' = \sum_{z=1}^{\infty} P(Z=z;\theta)\int_0^{\infty} z\alpha\beta^2 x^{r+2\alpha-1}(1+\beta x^{\alpha})^{z-1} e^{-z\beta x^{\alpha}} dx.$$

Let $u = \beta x^{\alpha} \rightarrow du = \alpha\beta x^{\alpha-1}dx$, then

$$\mu_r' = \sum_{z=1}^{\infty} zP(Z=z;\theta)\int_0^{\infty}\left(\frac{u}{\beta}\right)^{\frac{r}{\alpha}} u(1+u)^{z-1}e^{-uz}du.$$

Expanding it using binomial series and gamma function then we have the form

$$\mu_r' = \sum_{z=1}^{\infty}\sum_{i=0}^{z-1}\binom{z-1}{i}\frac{a_z\theta^z\Gamma\left(\frac{r}{\alpha}+i+1\right)}{K(\theta)z^{\frac{r}{\alpha}+i}\beta^{\frac{r}{\alpha}}},\quad r=1,2.... \tag{9}$$

Skewness (SK) and kurtosis (K) can be obtained from following relations respectively

$$SK = \frac{\mu'_3-3\mu'_2\,\mu'_1+2\mu'^3_1}{(\mu'_2-\mu'^2_1)^{\frac{3}{2}}},\qquad K = \frac{\mu'_4-4\mu'_3\,\mu'_1+6\mu'_2\,\mu'^2_1-3\mu'^4_1}{(\mu'_2-\mu'^2_1)^2},$$

where, μ_1', μ_2', μ_3' and μ_4' can be obtained from (9), by substituting $r=1,2,3,4.$

Also, *the mgf* about origin, $M_X(t)$, is defined

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r',$$

where, μ_r' is the rth raw moment. And then by using (9), the *mgf* of *GMEPS* is as follows:

$$M_X(t) = \sum_{z=1}^{\infty}\sum_{i=0}^{z-1}\frac{\binom{z-1}{i}a_z\theta^z\Gamma\left(\frac{r}{\alpha}+i+1\right)t^r}{M(\theta)z^{\frac{r}{\alpha}+i}\beta^{\frac{r}{\alpha}}\,r!},\quad r=1,2....$$

3.4 Order statistics

We obtain here the expression of probability density function of ith order statistics from the GME power series distribution. We use this expression to find the probability density functions of the lowest and largest order statistics.

. Let $Y_1 < Y_2 < ... < Y_n$ be the order statistics from the sample of size n. The pdf of $Y_i = X_{i:n}, i=1,...n$ is of the form

$$f_{i:n}(x;\Theta) = \frac{\Gamma(n+1)}{\Gamma(i)\,\Gamma(n-i+1)}[1-F(x;\Theta)]^{n-i}[F(x;\Theta)]^{i-1}f(x;\Theta), \tag{10}$$

where, $\Gamma(n)$ is the gamma function. By using cdf (5) and applying the binomial expansion we have

$$f_{i:n}(x;\psi)=\frac{\Gamma(n+1)\,f(x;\Theta)}{\Gamma(i)\,\Gamma(n-i+1)}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^{j}\left(\frac{M\left(\bar{G}(x;\alpha,\beta)\right)}{M(\theta)}\right)^{n+j-i}.$$

The expansion for expression $\left(M\left(\theta H(x)\right)\right)^{n+j-i}$ is obtained as

$$\left(M\left(\theta H(x)\right)\right)^{n+j-i}=\left(\sum_{z=1}^{\infty}a_z\theta^{z}e^{-(z-1)\beta x^{\alpha}}\bar{G}(x;\alpha,\beta)\right)^{n+j-i},$$

$$\left(M\left(\theta\bar{G}(x;\alpha,\beta)\right)\right)^{n+j-i}=\left(a_1\theta\bar{G}(x;\alpha,\beta)\right)^{n+j-i}\times$$
$$\left[1+\frac{a_2}{a_1}\theta\bar{G}(x;\alpha,\beta)+\frac{a_3}{a_2}\theta^{2}\left(\bar{G}(x;\alpha,\beta)\right)^{2}+\ldots\right]^{n+j-i}.$$

Hence,

$$\left(M\left(\bar{G}(x;\alpha,\beta)\right)\right)^{n+j-i}=a_1^{\,n+j-i}\times$$
$$\left(\sum_{m=0}^{\infty}\ell_m\left(\theta e^{-\beta x^{\alpha}}\left(1+\beta x^{\alpha}\right)^{m}\right)\right)^{n+j-i},\ell_m=\frac{a_{m+1}}{a_1},\ m=1,2,\ldots \qquad (11)$$

using the result from table of integral and series

$\left(\sum_{m=0}^{\infty}\ell_m Y^{m}\right)^{n+j-i}=\sum_{m=0}^{\infty}d_{n+j-i,m}Y^{m}$. Gradshteyn and Ryzhik (2000)

Then (11) implies as

$$\left(K\left(\left(\theta\bar{G}(x;\alpha,\beta)\right)\right)\right)^{n+j-i}=(a_1)^{n+j-i}\sum_{m=0}^{\infty}d_{n+j-i,m}\left(\theta\bar{G}(x;\alpha,\beta)\right)^{n+j-i+m}, \qquad (12)$$

where, $d_{n+j-i,0}=1$ and the coefficients $d_{n+j-i,m}$ are calculated from the following recurrence equation

$$d_{n+j-i,t}=t^{-1}\sum_{m=1}^{t}[m(n+j-i+1)-t]\ell_m d_{n+j-i,t-m},t\geq 1.$$

In addition,

$$M'\left(\theta\bar{G}(x;\alpha,\beta)\right)=\sum_{z=1}^{\infty}z\,a_z\left(\theta\bar{G}(x;\alpha,\beta)\right)^{z-1}.$$

Let $k=z-1$, then the above equation can be written as

$$M'\left(\theta\bar{G}\left(x;\alpha,\beta\right)\right)=\sum_{k=0}^{\infty}\ell_k(k+1)\left(\theta\bar{G}\left(x;\alpha,\beta\right)\right)^k,\ \ell_k=\frac{a_{k+1}}{a_1} \tag{13}$$

After replacing expansions (12) and (13) in equation (10) We have the following expression of the ith order statistic as

$$f_{i:n}(x;\Theta)=\frac{\theta g\left(x;\alpha,\beta\right)\sum_{k=0}^{\infty}\ell_k(k+1)\left(\theta\bar{G}\left(x;\alpha,\beta\right)\right)^k}{\mathrm{B}(i,n-i+j)(M(\theta))^{n+j-i+1}}$$
$$\times\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j a_1^{n+j-i+1}\sum_{m=0}^{\infty}d_{n+j-i,m}\left(\theta\bar{G}\left(x;\alpha,\beta\right)\right)^{n+j-i+m}.$$

Hence finally the expression of ith order statistic is:

$$f_{i:n}(x;\Theta)=\frac{\beta^2\alpha x^{2\alpha-1}}{\mathrm{B}(i,n-i+j)}\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}(-1)^j\binom{i-1}{j}\ell_k(k+1)$$
$$\times\frac{d_{n+j-i,m}a_1^{n+j-i+1}\theta^{n+j-i+m+k+1}e^{-(n+j-i+m+1+k)\beta x^\alpha}}{(M(\theta))^{n+j-i+1}}\left(1+\beta x^\alpha\right)^{n+j-i+m+k},\ x>0.$$

or

$$f_{i:n}(x;\Theta)=\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\tau_{j,k,m}\beta x^{2\alpha-1}\left(1+\beta x^\alpha\right)^{n+j-i+m+k}e^{-(n+j-i+m+k+1)\beta x^\alpha},\quad \text{where,}$$

$$\tau_{j,k,m}=(-1)^j\binom{i-1}{j}\frac{\alpha\lambda\ell_k(k+1)\theta^{n+j-i+m+k+1}a_1^{n+j-i+1}d_{n+j-i,m}}{\mathrm{B}(i,n-i+j)(M(\theta))^{n+j-i+1}}.$$

when we use binomial expansion we have another form of *pdf* as: :

$$f_{i:n}(x;\Theta)=\beta\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\sum_{h=0}^{n+j-i+m+k}\eta_{j,k,m,h}x^{\alpha(h+1)}e^{-(n+j-i+m+k+1)\beta x^\alpha}, \tag{14}$$

where,

$$\eta_{j,k,m,h}=(-1)^j\binom{i-1}{j}\binom{m+n+j-i+k}{h}\frac{\alpha\beta^{h+1}\theta^{n+j-i+m+k+1}\ell_k(k+1)a_1^{n+j-i+1}d_{n+j-i,m}}{\mathrm{B}(i,n-i+j)(M(\theta))^{n+j-i+1}}.$$

Now we obtain the pdfs of lowest order statistics and highest order statistics by replacing $i=1,n$, in (14), respectively, and expressions of both are as follows

$$f_{1:n}(x;\Theta)=\sum_{k=0}^{\infty}\sum_{`m=0}^{\infty}\sum_{h=0}^{n+j-i+m+k}\phi_{k,m,h}\beta x^{\alpha(h+1)}e^{-(n+m+k)\beta x^{\alpha}},$$

$$\phi_{k,m,h}=\binom{m+n+-1+k}{h}\frac{n\alpha\beta^{h+1}\ell_{k}(k+1)\theta^{n+m+k}a_{1}^{n}d_{n-1,m}}{(M(\theta))^{n}}.$$

and , $f_{n:n}(x;\Theta)=\sum_{k=0}^{\infty}\sum_{j=0}^{n-1}\sum_{`m=0}^{\infty}\sum_{h=0}^{j+m+k}\varsigma_{j,k,m,h}\beta x^{\alpha(h+1)}e^{-(j+m+k+1)\beta x^{\alpha}}$,

where,

$$\varsigma_{k,m,h}=\binom{m+j+k}{h}\binom{n-1}{j}(-1)^{j}\frac{n\beta^{h+1}\alpha\ell_{k}(k+1)\theta^{j+m+k+1}a_{1}^{j+1}d_{j,m}}{(M(\theta))^{j+1}}.$$

3.5 Rényi Entropy $I_R(x)$

The Rényi entropy $I_R(x)$ is a general form of Shannon entropy. Rényi entropy is used in such uncertainty where the other uncertainty measures like Shannon entropy are not suitable. The $I_R(x)$, for $\rho>0$, and $\rho\neq 1$, is defined as

$$I_R(x)=(1-\rho)^{-1}\log_b\left(\int_0^{\infty}\left(f(x;\Theta)\right)^{\rho}dx\right).$$

Let, $IP=\int_0^{\infty}\left(f(x;\Theta)\right)^{\rho}dx,$ then IP can be written as follows:

$$IP=\int_0^{\infty}\left(\theta g\left(x;\alpha,\beta\right)\right)^{\rho}\left\{\frac{M'\left(\theta\bar{G}\left(x;\alpha,\beta\right)\right)}{M(\theta)}\right\}^{\rho}dx.$$

But

$$\left(M'\left(\theta\bar{G}\left(x;\alpha,\beta\right)\right)\right)^{\rho}=a_1^{\rho}\left(\sum_{m=0}^{\infty}\delta_m\left(\theta\bar{G}\left(x;\alpha,\beta\right)\right)^{m}\right)^{\rho},\delta_m=\frac{a_{m+1}}{a_1},\ m=1,2,...$$

$$\left(\sum_{z=1}^{\infty}\delta_m\left(\theta\bar{G}\left(x;\alpha,\beta\right)\right)^{m}\right)^{\rho}=\sum_{m=0}^{\infty}d_{\rho,m}\left(\theta\bar{G}\left(x;\alpha,\beta\right)\right)^{m}.$$ [See Gradshteyn and Ryzhik (2000)]

Therefore,

$$\left(M'\left(\theta\bar{G}\left(x;\alpha,\beta\right)\right)\right)^{\rho}=a_1^{\rho}\sum_{z=1}^{\infty}d_{\rho,m}\left(\theta\bar{G}\left(x;\alpha,\beta\right)\right)^{m}. \quad (15)$$

Using the following coefficients for $t>1$ and they are computed from the following recurrence equation:

$$d_{\rho,t}=t^{-1}\sum_{m=1}^{t}[m(\rho+1)-t]\delta_m d_{\rho,t-m},d_{\rho,0}=1$$

Using binomial expansion for $\left(1+\lambda x^{\alpha}\right)^{m}$, then (15) will be as follows:

$$\left(M'\left(\theta\bar{G}\left(x;\alpha,\beta\right)\right)\right)^{\rho}=a_1^{\rho}\sum_{z=1}^{\infty}\sum_{k=0}^{m}\binom{m}{k}d_{\rho,m}\theta^{m}e^{-m\beta x^{\alpha}}\left(\beta x^{\alpha}\right)^{k}$$

Then the IP can be rewritten as follows

$$IP=\int_0^{\infty}\left(\alpha\beta\theta x^{\alpha-1}a_1\right)^{\rho}(1+\beta x^{\alpha})^{\rho}\sum_{m=0}^{\infty}\sum_{k=0}^{m}d_{\rho,m}\theta^{m}\binom{m}{k}(\beta x^{\alpha})^{k}e^{-(m+\rho)\beta x^{\alpha}}dx,$$

$$=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{h=0}^{\rho}\binom{m}{k}\binom{\rho}{h}d_{\rho,m}\theta^{m}\int_0^{\infty}\left(\alpha\beta\theta x^{\alpha-1}a_1\right)^{\rho}(\beta x^{\alpha})^{k+h}e^{-(m+\rho)\beta x^{\alpha}}dx.$$

After some simplification, then the Rényi entropy takes the following form

$$I_R(x)=(1-\rho)^{-1}\log_b\left[\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{h=0}^{\rho}\binom{m}{k}\binom{\rho}{h}\frac{d_{\rho,m}\theta^{m+\rho}\alpha^{\rho-1}a_1^{\rho}\Gamma(\frac{\rho(\alpha-1)+1}{\alpha}+k+h)}{(K(\theta))^{\rho}(m+\rho)^{\frac{\rho(\alpha-1)+1}{\alpha}+k+h}}\right]. \quad (16)$$

4. Reduced models

Some reduced models from *GMEPS* family of distributions for selected values of the parameters are presented in this section. Also, some sub-models; which are the generalized moment exponential Poisson distribution and moment exponential Poisson distribution are discussed in more details. Here we discuss some reduced models as:

1. For $M(\theta)=e^{\theta}-1$, (4) is a *GMEP* distribution with cdf:

$$F(x;\Theta)=\frac{e^{\theta}-\exp\left[\theta\bar{G}(x;\alpha,\beta)\right]}{e^{\theta}-1}, \qquad x,\alpha,\lambda,\beta>0. \qquad (17)$$

2. For $M(\theta)=e^{\theta}-1, \alpha=1,$ (4) is an *MEP* distribution with cdf:

$$F(x;\beta,\theta)=\frac{e^{\theta}-\exp\left[\theta H(x;\beta)\right]}{e^{\theta}-1}, \qquad x,\beta,\theta>0.$$

3. For $M(\theta)=-\ln(1-\theta)$, (4) is an G*MEL* distribution with cdf:

$$F(x;\Theta)=1-\frac{\ln\left[1-\theta\bar{G}(x;\alpha,\beta)\right]}{\ln(1-\theta)}, \qquad x,\beta,\alpha>0, \quad 0<\theta<1.$$

$$f(x)=\frac{\theta(2/\beta+1)g(x;\alpha,\beta)}{\ln(1-\theta)\left(1-\theta\bar{G}(x;\alpha,\beta)\right)}$$

4. For $M(\theta)=-\ln(1-\theta), \alpha=1,$ (4) is the *MEL* distribution with cdf:

$$F(x;\beta,\theta)=1-\frac{\ln\left[1-\theta H(x;\beta)\right]}{\ln(1-\theta)}, \qquad x,\beta>0, \quad 0<\theta<1.$$

5. For $M(\theta)=\theta(1-\theta)^{-1}$, (4) is the generalized *MEG* distribution with cdf:

$$F(x;\Theta)=\frac{G(x;\alpha,\beta)}{1-\theta\bar{G}(x;\alpha,\beta)}, \qquad x,\beta,\alpha>0, \quad 0<\theta<1.$$

6. For $M(\theta)=\theta(1-\theta)^{-1}, \alpha=1$ (4) is the *MEG* distribution with cdf:

$$F(x;\beta,\theta)=\frac{\bar{H}(x;\beta)}{1-\theta H(x;\beta)}, \qquad x,\beta>0, \ 0<\theta<1.$$

7. For $M(\theta)=(1-\theta)^{m}-1,$ (4) is the *MEB* distribution with cdf:

$$F(x;\Theta)=\frac{(1-\theta)^{m}-\left[1-\theta\bar{G}(x;\alpha,\beta)\right]^{m}}{(1-\theta)^{m}-1}, \qquad x,\beta,\alpha>0, \quad 0<\theta<1.$$

4.1 GME Poisson distribution

GMEP distribution is a reduced model of *GMEPS'* distribution. The pdf of the *GMEP* distribution corresponding to (17) is of the form

$$f(x;\Theta)=\frac{\theta\, g(x;\alpha,\beta)\exp\left(\theta\bar{G}(x;\alpha,\beta)\right)}{(e^{\theta}-1)} \quad ,x,\beta,\alpha,\theta>0. \qquad (18)$$

The reliability of GMEP distribution and hazard rate function of *GMEP* distribution have the expressions as:

$$R(x;\Theta)=\frac{\exp\left[\theta\bar{G}(x;\alpha,\beta)\right]-1}{e^{\theta}-1},$$

and $$h(x;\Theta)=\frac{\theta g(x;\alpha,\beta)\exp\left(\theta\bar{G}(x;\alpha,\beta)\right)}{\left[\exp\left(\theta\bar{G}(x;\alpha,\beta)\right)-1\right]}.$$

Figure 1 and Figure 2 discuss the behavior of PDF of GMEP distribution and hazard rate function for parameter values.

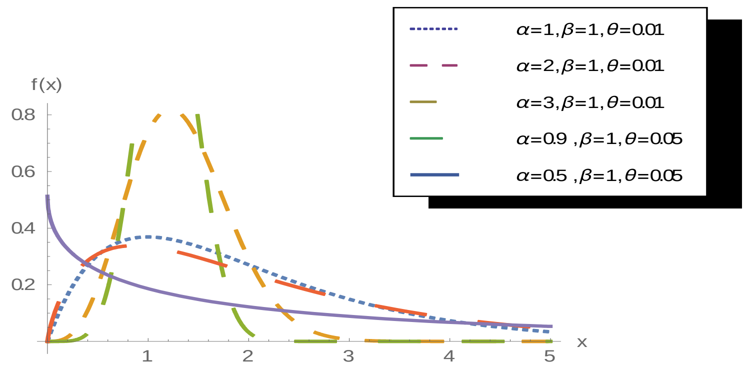

Figure 1.The pdf plots of the *GMEP* distribution

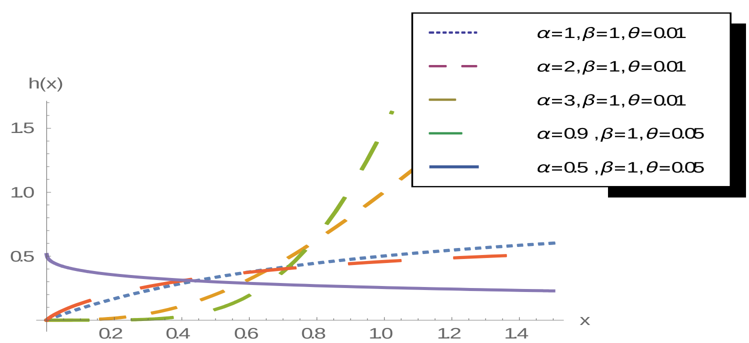

Figure 2. The hazard rate plots for the *GMEP* distribution

Figure 2 provides increasing, decreasing and constant failure rates of *GMEP* distribution. The $Q(p)$ for the *GMEP* distribution can be found from (8) as

$$(Q(p))^{\alpha} = -\frac{1}{\lambda} - W[-\frac{\ln(p+(1-p)e^{\theta})}{\theta e^{1}}].$$

Solving this equation for $Q(p)$, the quantile function of *GMEP* is obtained.

Furthermore, the rth moment of the *GMEP* distribution about zero is given by

$$P(Z = z;\theta) = \frac{e^{-\theta}\theta^{z}}{z!(1-e^{-\theta})},\ z = 1,2,...$$

in (9) as follows

$$\mu_r' = \sum_{z=1}^{\infty}\sum_{j=0}^{z-1}\sum_{i=0}^{j+1}\binom{z-1}{j}\binom{j+1}{i}\frac{\theta^{z}\Gamma\left(\frac{r}{\alpha}+i+1\right)}{z!(e^{\theta}-1)z^{\frac{r}{\alpha}+i}\lambda^{\frac{r}{\alpha}}},$$

$$r = 1,2....$$

Additionally the Rényi entropy is obtained by substituting $K(\theta) = e^{\theta} - 1$, in (16) as follows

$$I_R(x) = (1-\rho)^{-1}\log_b\left[\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{h=0}^{\rho}\binom{m}{k}\binom{\rho}{h}\frac{d_{\rho,m}\theta^{m+\rho}\alpha^{\rho-1}a_1^{\rho}\Gamma(\frac{\rho(\alpha-1)+1}{\alpha}+k+h)}{(e^{\theta}-1)^{\rho}(m+\rho)^{\frac{\rho(\alpha-1)+1}{\alpha}+k+h}}\right].$$

4.2 GME geometric distribution

GMEG distribution is a member of *GMEPS* family of distribution as a special case. The pdf of the *GMEG* distribution corresponding to (18) is of the form

$$f(x;\Theta) = \frac{g(x;\alpha,\beta)(1-\theta)}{\left[1-\left(\theta\bar{G}(x;\alpha,\beta)\right)\right]^{2}}, \quad x > 0, 0 < \theta < 1, \alpha,\beta > 0.$$

(19)

The expressions of reliability function $R(x;\Theta)$ and hazard rate function $h(x;\Theta)$ are:

$$R(x;\Theta) = \frac{(1-\theta)\bar{G}(x;\alpha,\beta)}{1-\theta\bar{G}(x;\alpha,\beta)},$$

and,

$$h(x;\Theta)=\frac{g\left(x;\alpha,\beta\right)}{\bar{G}\left(x;\alpha,\beta\right)\left[1-\left(\theta\bar{G}\left(x;\alpha,\beta\right)\right)\right]}.$$

Figures 3 and 4 represent *pdf* and *hrfs* plots for *GMEG* distribution for some selected values of parameters.

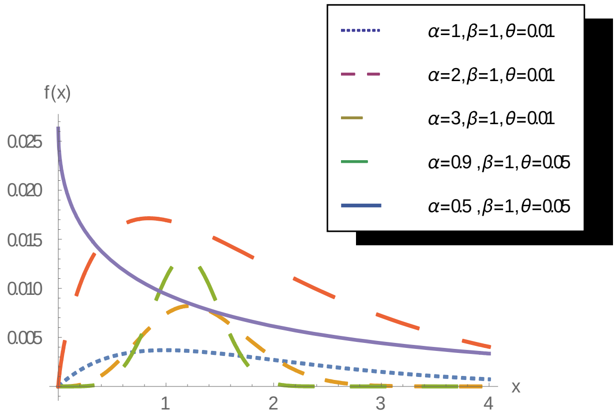

Figure.3. The pdf plots of the *GMEG* distribution

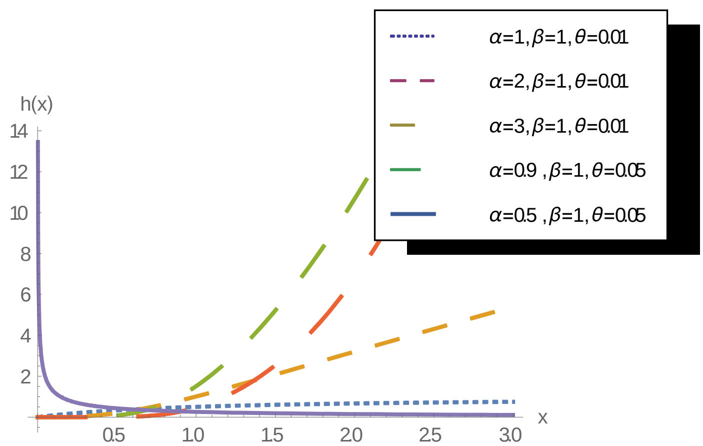

Figure. 4. The hazard rate plots of the *GMEG* distribution

It is observed that the shapes of the *hrf* are decreasing increasing bathtub shape, decreasing, increasing and constant at some parameter values.

The $Q(p)$ function for the *GMEG* distribution is as:

$$(Q(p))^{\alpha} = -\frac{1}{\lambda} - W[-\frac{(1-p)}{(1-\theta p)e^{1}}].$$

Solving this equation for $Q(p)$, for different values of p.
The rth moment about zero can be obtained by
$P(Z=z;\theta)=(1-\theta)\theta^{z-1}, \ z=1,2,...,$ in (9) as follows

$$\mu_r' = \sum_{z=1}^{\infty}\sum_{j=0}^{z-1}\sum_{i=0}^{j+1}\binom{z-1}{j}\binom{j+1}{i}\frac{\theta^{z-1}(1-\theta)\Gamma\left(\frac{r}{\alpha}+i+1\right)}{z^{\frac{r}{\alpha}+i}\lambda^{\frac{r}{\alpha}}}, \quad r=1,2.... \qquad (20)$$

Further, the Rényi entropy is obtained by substituting $M(\theta)=\theta(1-\theta)^{-1}$, in (16) as follows

$$I_R(x) = (1-\rho)^{-1}\log_b\left[\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{h=0}^{\rho}\binom{m}{k}\binom{\rho}{h}\frac{d_{\rho,m}\theta^m\lambda^{\rho+h+k}\alpha^{\rho-1}(1-\theta)^{\rho}a_1^{\rho}\Gamma(\frac{\rho(\alpha-1)+1}{\alpha}+k+h)}{(m+\rho)^{\frac{\rho(\alpha-1)+1}{\alpha}+k+h}}\right].$$

5. Parameter estimation of the *GMEPS* family

In this section, parameters' estimation of the parameters is conducted through maximum likelihood method.

Let $X_1, X_2, ...X_n$ be a simple random sample from the *GMEPS* family with set of parameters $\Theta \equiv (\alpha,\beta,\theta)$. The log-likelihood function based on the observed random sample of size n is given by:

$$f(x;\Theta) = \frac{\theta\, g(x;\alpha,\beta)M'\left(\theta\,\bar{G}(x;\alpha,\beta)\right)}{M(\theta)}, x,\beta,\alpha,\theta,>0.$$

$$f(x;\Theta) = \frac{\theta\,\alpha\beta^2x^{2\alpha-1}e^{-\beta x^{\alpha}}M'\left(\theta\,\bar{G}(x;\alpha,\beta)\right)}{M(\theta)}, x,\beta,\alpha,\theta,>0.$$

$$L(x;\Theta) = \alpha^n\beta^{2n}\theta^n\left(\prod_{i=1}^{n}x\right)^{2\alpha-1} e^{-\beta\sum_{i=1}^{n}x^{\alpha}}\frac{\prod_{i=1}^{n}M'\left(\theta\bar{G}(x;\alpha,\beta)\right)}{\left(M(\theta)\right)^n}$$

$$\ln L(x;\Theta) = n\ln\alpha + 2n\ln\beta + n\ln\theta + \left(2\ \alpha - 1\right)\sum_{i=1}^{n}\ln x_i - \beta\ \sum_{i=1}^{n} x_i^{\alpha}$$

$$+\sum_{i=1}^{n}\ln\left(M'(\theta\bar{G}\left(x;\alpha,\beta\right))\right) - n\ln(M(\theta)).$$

where, $\ln L = \ln L(x;\Theta)$. The partial derivatives of the log-likelihood function w.r.t the parameters:

$$\frac{\partial \ln L}{\partial\alpha} = \frac{n}{\alpha} + 2\sum_{i=1}^{n}\ln x_i - \beta\sum_{i=1}^{n} x_i^{\alpha}\ln x_i - \theta\sum_{i=1}^{n}\frac{M''(\theta\bar{G}\left(x;\alpha,\beta\right))}{M'(\theta\bar{G}\left(x;\alpha,\beta\right))}\frac{\partial\bar{G}\left(x;\alpha,\beta\right)}{\partial\alpha},$$

$$\frac{\partial \ln L}{\partial\beta} = \frac{2n}{\beta} - \sum_{i=1}^{n} x_i^{\alpha} + \theta\sum_{i=1}^{n}\frac{M''(\theta\bar{G}\left(x;\alpha,\beta\right))}{M'(\theta\bar{G}\left(x;\alpha,\beta\right))}\frac{\partial\bar{G}\left(x;\alpha,\beta\right)}{\partial\beta},$$

$$\frac{\partial \ln L}{\partial\theta} = \frac{n}{\beta} + \theta\sum_{i=1}^{n}\left[\frac{M''(\theta\bar{G}\left(x;\alpha,\beta\right))}{M'(\theta\bar{G}\left(x;\alpha,\beta\right))}\right]\bar{G}\left(x;\alpha,\beta\right) - \frac{nM'(\theta)}{M(\theta)},$$

where,

$$\frac{\partial\bar{G}}{\partial\alpha} = -\beta^2 x_i^{2\alpha} e^{-\beta x_i^{\alpha}}\ln x_i, \quad \text{and,} \quad \frac{\partial\bar{G}}{\partial\beta} = -\lambda x_i^{2\alpha}.$$

The solution of equations $\frac{\ln L}{\partial\alpha} = 0, \frac{\ln L}{\partial\beta} = 0, \frac{\ln L}{\partial\theta} = 0$ through software will be the estimates of parameters.

5.1. A Simulation Study:

We use the Monte Carlo (MC) simulation to check the convergence of ML estimator's of $\hat{\underline{\Theta}}$ through Mathematica 10.2 version. We generate random sample of size n from the model of *GMEG* distribution. We find the ML estimates of the parameters for different sample sizes. The amount of bias with mean square error (*MSE*) under the repetition10000 is calculated for each sample. From table the amount of bias and MSE are decreases as sample sizes increases.

Table 1. The Bias and MSE on Monte Carlo simulation for parameters values for the *GMEG* model

Parameter	True value	Sample size n	Mean	Bias	MSE
α	2	$n=30$	2.2437	0.2437	1.0321
		$n=50$	2.2321	0.2321	0.9014
		$n=100$	2.2232	0.2232	0.7932
		$n=300$	2.1524	0.1524	0.5012
		$n=500$	2.0517	0.0517	0.3223
		$n=1000$	2.0039	0.0039	0.2015
β	3	$n=30$	3.2537	0.2537	0.9423
		$n=50$	3.2420	0.2420	0.8317
		$n=100$	3.2412	0.2412	0.7694
		$n=300$	3.2015	0.2015	0.7062
		$n=500$	3.1436	0.1436	0.4319
		$n=1000$	3.0219	0.0219	0.1726
θ	0.5	$n=30$	0.6813	0.1813	0.4536
		$n=50$	0.6801	0.1801	0.3998
		$n=100$	0.6521	0.1521	0.3457
		$n=300$	0.5523	0.0523	0.1929
		$n=500$	0.5176	0.0176	0.1612
		$n=1000$	0.5069	0.0069	0.0134

In table 2 we use the technique of method of moments to find the estimated interval of values for each parameter. We see that by increasing sample size we have larger amount of percentage for two specific values.

Table 2: Percentage of sample estimates of $\Theta=(\alpha,\beta,\theta)$ through method of moments (MM) for the *GMEG* model

N	% estimated values for $\alpha=2$	% estimated values for $\beta=3$	% estimated values $\theta=0.5$
	$1.4<\hat{\alpha}<2.6$	$2.5<\hat{\beta}<3.5$	$0.3<\hat{\theta}<0.7$
30	87.58%	86.18%	80.02%
50	93.04%	90.26%	85.52%
100	97.35%	93.94%	88.71%
250	98.92%	97.42%	94.56%
500	99.59%	99.01%	96.69%
1000	99.86%	99.45%	98.94%

6. APPLICATIONS

In this section, we apply the special models of GMEPS to two real data set and check its flexibility.

Murthy et al. (2004, p.297) used data set related failure times of 84 model aircraft windshield with unit of measurement is 1000 h. The data are : 4.602, 1.757,2.324, 3.376, 4.663,1.619,2.224, 2.688, 3.924,1.505, 2.154, 2.964 1.303, 2.089, 2.902, 4.278,2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 2.661, 3.779, 1.248, 2.010, 2.223, 3.114, 4.449, 2.962, 4.255, 3.117, 4.485, 1.652 4.167, 1.432,2.097, 2.934, 4.240, 1.480, 2.135,0.040, 1.070,1.914, 2.646, 1.866, 2.385, 3.443, 3.467, 0.309,1.899, 2.610, , 2.229, 3.166, 4.570, 1.652, 1.506,2.190, 3.000, 3.103, 4.376, 1.615,2.300, 3.478, 0.557, 1.911, 2.625, 1.281,2.038, 4.305, 1.568, 1.981, , 2.194, 3.578, 0.943, 1.912, 2.632, 3.595, 0.301, 1.876, 2.481, 3.699, 1.124, 3.344.

We estimate unknown parameters of the GMEG distribution by ML method as describe in section 5 by using the R code. We calculate the value of Kolmogorov Smirnov test statistics and some other measures for goodness. We see that *GMEG* distribution proves better fit than other models shown in the following table.

Table 3. Criteria for comparison for second data set

Model	k-s	- Log L	*AIC*	*CAIC*	*BIC*
GMEG	0.681	123.79	263.58	195.89	268.96
WD	0.742	128.05	264.10	205.06	270.87
EE	0.721	137.84	283.68	227.93	288.54
E	0.694	161.88	327.75	218.85	330.18

Smaller values of these statistics indicate a better fit.
k-s denotes Kolmogorov- Smirnov test statistic

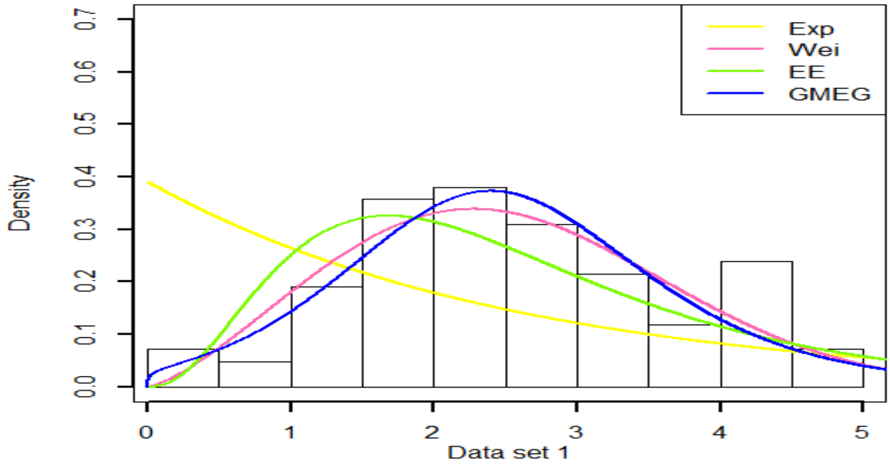

Figure 5. Estimated densities of models for the second data set

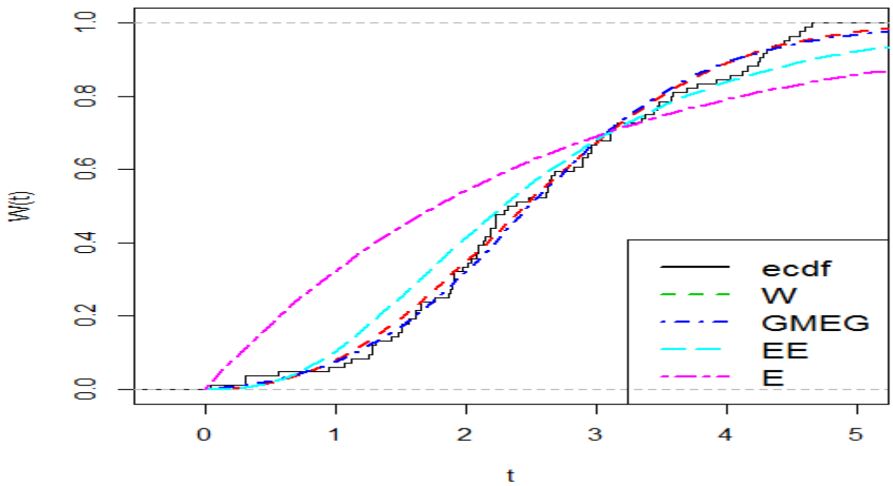

Figure 6 Estimated cumulative densities of models for the first data set

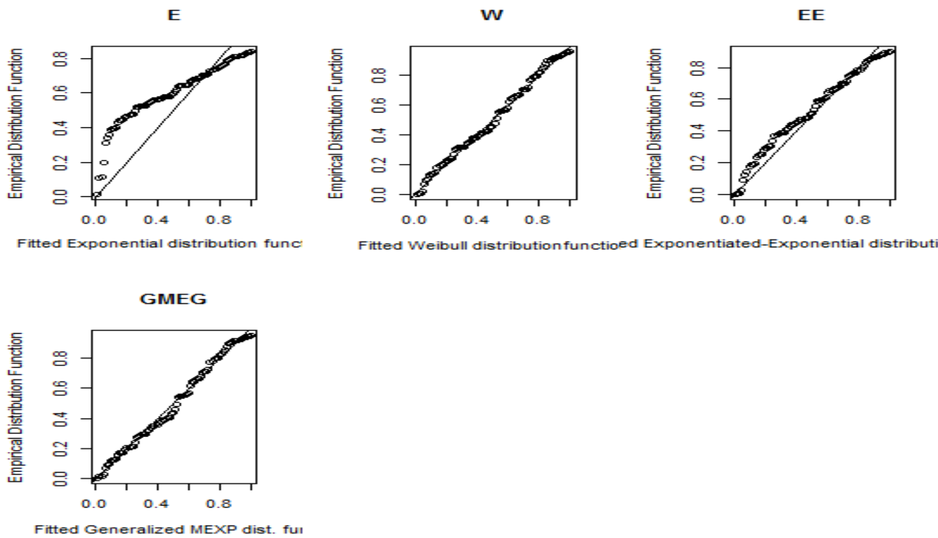

Figure 7: The probability–probability plots for the aircraft windshield data set.

The second data set related to the survival times (measured in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The data are as follows:

1.09, 1.83, 2.3,1.15, 2.53, 2.54, 2.78, 1.12, 1.63, 1.97, 1.46, 2.02, 2.13, 5.55,2.15, 1.96, 1.53, 1.59, 1.6, 1.36, 2.54, 0.77, 1.3, 1.34, 1, 1.02, 0.72,1.08, 1.21, 1.22, 1.68, 0.44,1.13, 0.1, 0.33, 2.93, 2.31, 2.4, 2.45, 2.51, 3.42, 3.47, 1.08, 1.22, 3.27, 1.16, 1.2, 0.59, 0.59,1.08, 1.39, 1.44, 1.95, 1.07,2.16,1.24, 2.22, 0.74, 0.92, 1, 1.71, 1.72, 1.76, 0.56, 4.58, 1.05, 1.07,3.61, 4.02, 4.32, 0.93, 0.96.

Table 4. Criteria for comparison for 2nd data set

Model	*k-s*	-Log L	AIC	CAIC	BIC
GMEG	0.823	88.765	193.53	193.87	200.34
WD	0.832	94.03	196.06	196.22	200.60
EE	0.853	93.475	194.95	195.33	201.50
E	0.844	109.45	226.89	226.95	229.16

For the second data set, the values of *k-s*, *AIC*, *BIC* and *CAIC* are recorded in table 5. The plots of the estimated densities are shown in Figure 8.

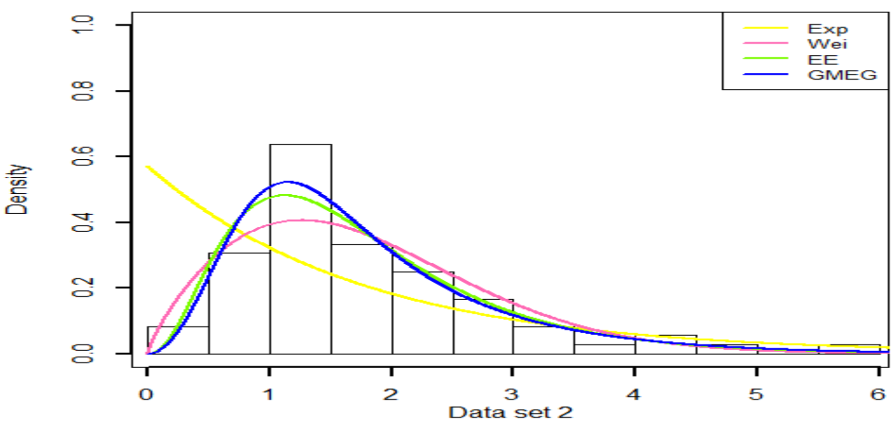

Figure 8. Estimated densities of models for the data set

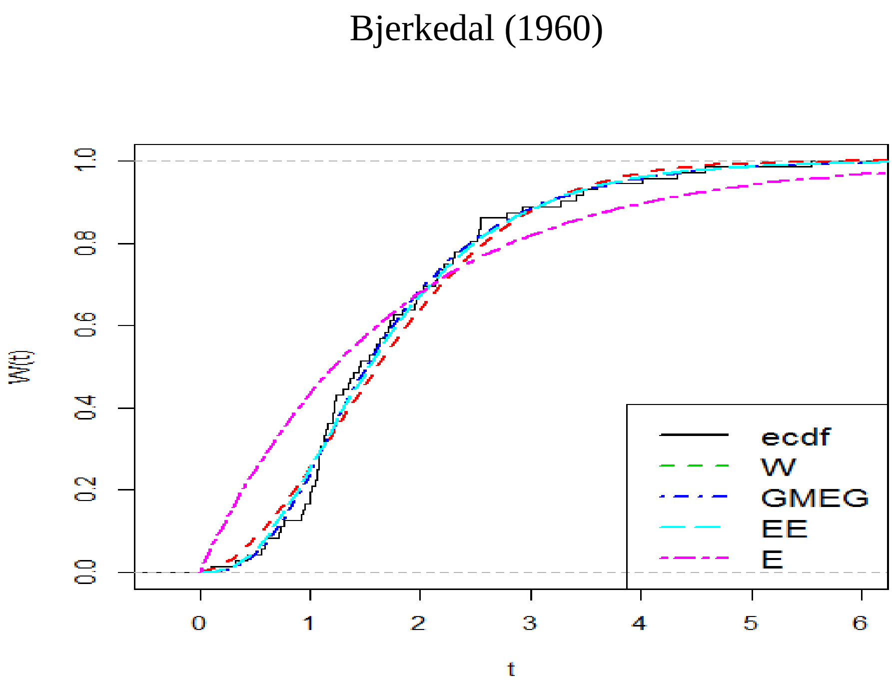

Figure. 9. Estimated cumulative densities of models for the second data set

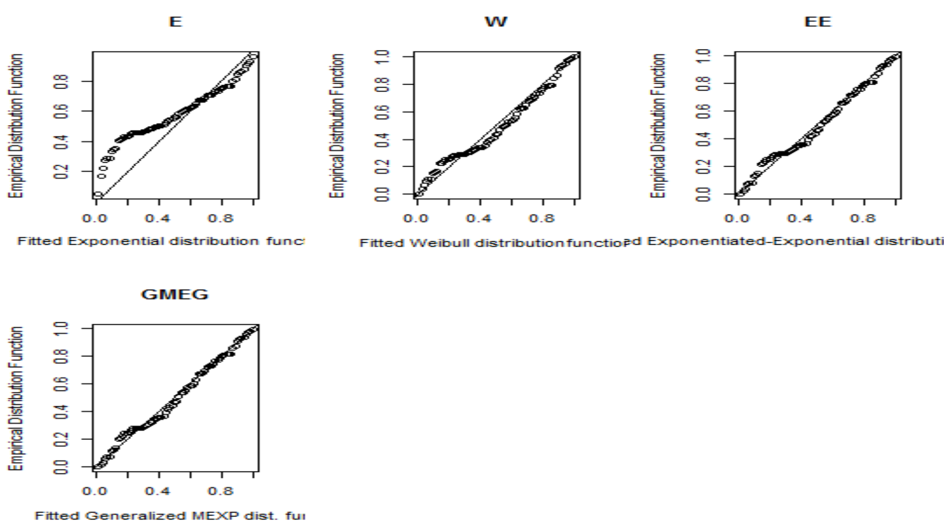

Figure 10: The probability–probability plots for the Bjerkedal (1960) data set

We observe from the table values and graphs that the new GMEG provides better fit than other models.

7. Concluding Remarks

In this research paper we develop a new family for lifetime data. This model is generated through compound technique. The GMEPS is compounded through the GME distribution and truncated power series distribution**.** We have shown a number of sub-models of GMEPS distribution which indicate its flexibility. We have derived some statistical properties of this new distribution. The hazard rate functions of sub-models have various shapes such as decreasing, increasing, and bathtub. The amount of bias and MSE approach to zero when sample size tends to indefinitely large. The related model GMEG distribution associated to this family are applied on two real data sets. The new family proves better fit than some of existing models available in literature.

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