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Original Research Article

Relative Extensions of Maps

Relative Extension of Continuous Mappings

this paper

ABSTRACT. In the article, the notion of a relative extension of continuous mappings is defined. The relative extension of continuous mappings is the generalization of the notion of a relative retract in topological spaces. They have been applied to the fixed point theory.

Keywords: extension space, neighborhood extension space, movable space, space of trivial shape, relative extension space, relative neighborhood extension space, fixed point, \mathbb{D} -retract.

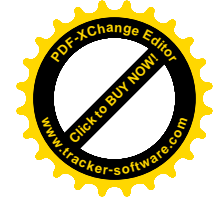
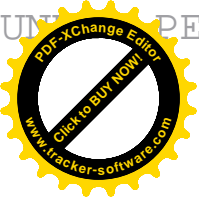
1 Introduction

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In the article [9] we have introduced the notion of a relative retract in metric spaces and defined the class of absolute relative retracts (ARR) and absolute neighborhood retract ($ANRR$). The relative retracts are an essential generalization of the retracts in the sense of Borsuk. In articles [9, 10, 11] their properties are studied with the use of new topological tools (relative homotopy, relative contractability). Relative retracts are applied to fixed point theory and the theory of coincidence (see [9, 10]) as well as applied to the study of global and local properties of metrizable spaces (see [11]). In this article we define the relative extension of continuous maps, that is, the generalization of the notion of relative retracts in topological spaces (not necessarily metrizable). We also define two classes of topological spaces: relative extension (ESR) and relative neighborhood extension ($NESR$). These spaces are an essential generalization of spaces ES and NES respectively, considered by G. Fournier and A. Granas in the article [4]. The relative extension of continuous maps is applied to the theorems on the fixed points of multivalued compact and noncompact maps. This article is an integral part of relative retracts theory and relative homotopy and should be regarded in the context of [9, 10, 11].

Please reconstruct the last sentence.

I will suggest you write the full meaning of ES and NES , since it was mentioned the first time in this paper.



2 Preliminaries

provided

Throughout this paper, all spaces are assumed to be Hausdorff topological spaces and all singlevalued mappings are continuous. A continuous mapping $f : X \rightarrow Y$ is called perfect, if for every $y \in Y$, the set $f^{-1}(y)$ is nonempty and compact and f is a closed map. Let X and Y be two spaces and assume that for every $x \in X$ a nonempty subset $\varphi(x)$ of Y is given. In such a case we say that $\varphi : X \multimap Y$ is a multivalued mapping. Let H_* be the Čech homology functor with compact carriers and coefficients in the field of rational numbers \mathbb{Q} from the category of Hausdorff topological spaces and continuous maps to the category of a graded vector space and linear maps of degree zero. Thus $H_*(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being a q -dimensional Čech homology group with compact carriers of X . For a continuous map $f : X \rightarrow Y$, $H_*(f)$ is the induced linear map $f_* = \{f_q\}$ where $f_q : H_q(X) \rightarrow H_q(Y)$ ([5]). A set X is acyclic if:

In this case,

- (i) X is nonempty,
- (ii) $H_q(X) = 0$ for every $q \geq 1$ and
- (iii) $H_0(X) \approx \mathbb{Q}$.

Let $u : E \rightarrow E$ be an endomorphism of an arbitrary vector space. Let us put Suppose

$$N(u) = \{x \in E : u^n(x) = 0 \text{ for some } n\},$$

where u^n is the n -th iterate of u and $\tilde{E} = E/N(u)$. Since $u(N(u)) \subset N(u)$, we have the induced endomorphism $\tilde{u} : \tilde{E} \rightarrow \tilde{E}$ defined by $\tilde{u}([x]) = [u(x)]$. We call u admissible provided $\dim \tilde{E} < \infty$. Let $u = \{u_q\} : E \rightarrow E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call u a Leray endomorphism if

- (i) all u_q are admissible,
- (ii) almost all \tilde{E}_q are trivial.

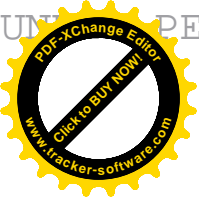
For such a u , we define the (generalized) Lefschetz number $\Lambda(u)$ of u by putting

$$\Lambda(u) = \sum_q (-1)^q \text{tr}(\tilde{u}_q),$$

where $\text{tr}(\tilde{u}_q)$ is the ordinary trace of \tilde{u}_q (comp. [5]). The following important property of a Leray endomorphism is a consequence of a well-known formula $\text{tr}(u \circ v) = \text{tr}(v \circ u)$ for the ordinary trace.

Proposition 2.1 [5] Assume that, in the category of graded vector spaces the following diagram commutes

$$\begin{array}{ccc} E' & \xrightarrow{u} & E'' \\ u' \uparrow & \swarrow v & \uparrow u'' \\ E' & \xrightarrow{u} & E'' \end{array}$$



If one of u' , u'' is a Leray endomorphism, then so is the other; and $\Lambda(u') = \Lambda(u'')$.

An endomorphism $u : E \rightarrow E$ of a graded vector space E is called weakly nilpotent if for every $q \geq 0$ and for every $x \in E_q$, there exists an integer n such that $u_q^n(x) = 0$.

Since, for a weakly nilpotent endomorphism $u : E \rightarrow E$, we have $N(u) = E$, we get:

Proposition 2.2. *For If $u : E \rightarrow E$ is a weakly nilpotent endomorphism, then $\Lambda(u) = 0$.*

A perfect map $p : X \rightarrow Y$ is called Vietoris, provided for every $y \in Y$, the set $p^{-1}(y)$ is acyclic. We recall that the composition of two Vietoris mappings is a Vietoris mapping and if $p : X \rightarrow Y$ is a Vietoris map then $p_* : H_*(X) \rightarrow H_*(Y)$ is an isomorphism (see [5]). Let $\varphi : X \multimap Y$ be a multivalued map. We recall that the map φ is admissible (s-admissible) (see [5]) if there exist a Vietoris map $p : Z \rightarrow X$ and a continuous map $q : Z \rightarrow Y$ such that for each $x \in X$

Are the two mappings the same?

$$q(p^{-1}(x)) \subset \varphi(x) \quad (q(p^{-1}(x)) = \varphi(x)) \quad (\text{we will write } (p, q) \subset \varphi \quad ((p, q) = \varphi)).$$

Let $\varphi : X \multimap Y$ be a map and let $A \subset X$ be a nonempty set. We denote $\varphi_A : A \multimap Y$ a map given by the formula $\varphi_A(x) = \varphi(x)$ for each $x \in A$.

Definition 2.3. *A topological vector space is called Klee admissible provided for every compact $K \subset E$ and for every open neighborhood of zero V in E there exists a continuous map $\pi_V : K \rightarrow E$ such that:*

(2.3.1) $(x - \pi_V(x)) \in V$ for every $x \in K$, and

(2.3.2) there exists a natural number $n = n_K$ such that $\pi_V(K) \subset E^n$, where E^n is an n -dimensional subspace of E .

It is well known that any locally convex space is Klee admissible. We recall that a multivalued map $\varphi : X \multimap Y$ is compact, if the set $\overline{\varphi(X)} \subset Y$ is compact.

Theorem 2.4. [5] *Let E be a Klee admissible space and let $U \subset E$ be an open set. Consider a diagram:*

$$U \xleftarrow{p} Z \xrightarrow{q} U,$$

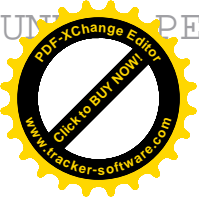
p is Vietoris and q is compact. Then $q_ \circ p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_* \circ p_*^{-1}) \neq 0$ implies that p and q have a coincidence point, that is, there is a point $z \in Z$ such that $p(z) = q(z)$.*

Theorem 2.5. [6] *Let X be normal, $A \subset X$ closed, and $F_0 : A \rightarrow E$ a compact map into a normed space E . Then F_0 is extendable to a compact map $F : X \rightarrow E$.*

Remark suppose to just be a statement, I will suggest you change it to Lemma or Proposition

Remark 2.6. *Let X be normal, $A \subset X$ closed, and $F_0 : A \rightarrow U$ a compact map into an open set $U \subset E$, where E is a normed space. Then F_0 is extendable to a compact map $F : V \rightarrow U$, where $V \subset X$ is some open neighborhood of A .*

Proof. Let $F_0 : A \rightarrow U$ be a compact map, that is, $\overline{F_0(A)} \subset U$ is a compact set, where $A \subset X$ is closed and $U \subset E$ is an open set. There exists an open neighborhood $V_1 \subset U$ of $\overline{F_0(A)}$ such that $\overline{V_1} \subset U$. From Theorem 2.5 there exists a compact extension $F : X \rightarrow E$ of $F_0 : A \rightarrow U \subset E$. Let $V = F^{-1}(V_1)$. We define a map $\tilde{F} : V \rightarrow U$ by the formula $\tilde{F}(x) = F(x)$ for each $x \in V$. We observe that the map \tilde{F} is a compact extension of F_0 and the proof is complete. \square



Definition 2.7. (see [4]) A space \mathbb{M} is an extension (resp. neighborhood extension) space provided for any compact pair (X, A) with $A \subset X$ closed and any map $f_0 : A \rightarrow \mathbb{M}$ there is an extension $f : X \rightarrow \mathbb{M}$ (resp. neighborhood extension $f : U \rightarrow \mathbb{M}$) of f_0 over X (resp. over the neighborhood of A in X). The classes of the extension spaces and the neighborhood extension spaces will be denoted by ES (written $\mathbb{M} \in ES$) and NES (written $\mathbb{M} \in NES$), respectively.

Let $X \in ANR$ and let $Y \subset X$ be a compact and nonempty subset. We recall that Y is movable in X provided every neighborhood U of Y admits a neighborhood V of Y , $V \subset U$, such that for every neighborhood W of Y , $W \subset V$, there exists a homotopy $H : V \times [0, 1] \rightarrow U$ with

$$(1) \quad H(x, 0) = x \quad \text{and} \quad H(x, 1) \in W, \quad \text{for any } x \in V.$$

Let $y_0 \in Y$. We recall that Y has a trivial shape in X provided every neighborhood U of Y admits a neighborhood V of Y , $V \subset U$, such that there exists a homotopy $H : V \times [0, 1] \rightarrow U$ with

$$(2) \quad H(x, 0) = x \quad \text{and} \quad H(x, 1) = y_0, \quad \text{for any } x \in V.$$

Let Y be a compact and metrizable space. We say that Y is movable (has a trivial shape) provided there exists a space $X \in ANR$ and an embedding $h : Y \rightarrow X$ such that $h(Y)$ is movable (has a trivial shape) in X .

Remark 2.8. We recall that in the metrizable spaces the property of movable (trivial shape) is an absolute property, that is, if a compact set Y is movable (of trivial shape) in some ANR X and $h : Y \rightarrow X'$ is an embedding into an ANR X' , then $h(Y)$ is movable (of trivial shape) in X' (see [1]).

Remark 2.9. [1] We know that movable spaces are of the following types, among others: AR , ANR , FAR and $FANR$.

The following are types of movable spaces: AR , ANR , FAR and $FANR$.

3 The families of sets

section, we will give the necessary notions through definitions and lemmas.

In this paragraph a few necessary notions will be defined.

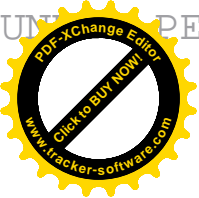
Definition 3.1 (Trivial shape). Let \mathbb{T} be a Tychonoff cube, $X \subset \mathbb{T}$ be a compact space and let $x_0 \in X$ be an arbitrary point. We will say that X has a trivial shape in \mathbb{T} provided every neighborhood $U \subset \mathbb{T}$ of X admits a neighborhood V of X , $V \subset U$, such that there exists a homotopy $H : V \times [0, 1] \rightarrow U$ with

$$H(x, 0) = x \quad \text{and} \quad H(x, 1) = x_0, \quad \text{for any } x \in V, \quad (\text{see } (2)).$$

A compact space X has a trivial shape if there exists a Tychonoff cube \mathbb{T} and an embedding $h : X \rightarrow \mathbb{T}$ such that $h(X)$ has a trivial shape in \mathbb{T} .

Proposition 3.2. Let S be a nonempty set and let $X = \prod_{s \in S} X_s$. Assume that for each $s \in S$ there exists a Tychonoff cube \mathbb{T}_s such that $X_s \subset \mathbb{T}_s$. The space X has a trivial shape in $\mathbb{T} = \prod_{s \in S} \mathbb{T}_s$ if and only if for each $s \in S$ a space X_s has a trivial shape in \mathbb{T}_s .

No reference. Is this a new proposition by you? If no, please let us know where you get it from. 4



Proof. Assume that X has a trivial shape in \mathbb{T} . Let $\lambda \in X$, $\lambda = \{\lambda_s\}$ and let $a \in S$. We show that X_a has a trivial shape in \mathbb{T}_a . Let $U_a \subset \mathbb{T}_a$ be a compact neighborhood of X_a . The set $U = \prod_{s \in S} U_s$, where $U_s = \mathbb{T}_s$ for each $s \neq a$ is a compact neighborhood of X in $\mathbb{T} = \prod_{s \in S} \mathbb{T}_s$. There exists a compact neighborhood $V = \prod_{s \in S} V_s \subset U$ of X , $V_s = \mathbb{T}_s$ for each $s \notin \{s_1, s_2, \dots, s_n, a\}$ such that there exists a homotopy $H_\lambda: V \times [0, 1] \rightarrow U$ with

$$H_\lambda(x, 0) = x \text{ and } H_\lambda(x, 1) = \lambda, \text{ for any } x \in V.$$

We have Suppose that

$$V_a \times [0, 1] \xrightarrow{h} V \times [0, 1] \xrightarrow{H_\lambda} U \xrightarrow{\pi} U_a,$$

where h is a homeomorphism given by the formula $h(v_a, t) = (\{y_s\}, t)$, $y_a = v_a$, $y_s = \lambda_s$ for $s \neq a$ and π is a projection. **We** define a homotopy $H_a: V_a \times [0, 1] \rightarrow U_a$ by the formula:

$$H_a = \pi \circ H_\lambda \circ h.$$

Assume now that, for each $s \in S$ a space X_s has a trivial shape in \mathbb{T}_s . Let $\lambda \in X$, $\lambda = \{\lambda_s\}$ be an arbitrary point and let $U = \prod_{s \in S} U_s \subset \mathbb{T}$ be a compact neighborhood of X in \mathbb{T} such that for each $s \notin \{s_1, s_2, \dots, s_n\}$, $U_s = \mathbb{T}_s$. **From the assumption** for each $k = 1, \dots, n$ there exists a compact neighborhood $V_{s_k} \subset U_{s_k}$ of X_{s_k} and a homotopy $H_{s_k}: V_{s_k} \times [0, 1] \rightarrow U_{s_k}$ with

It follows that,

$$H_{s_k}(x, 0) = x \text{ and } H_{s_k}(x, 1) = \lambda_{s_k}, \text{ for any } x \in V_{s_k}.$$

Please check, I guess the result should be s_k .

Let $V = \prod_{s \in S} V_s$, $V_s = \mathbb{T}_s$ for any $s \notin \{s_1, s_2, \dots, s_n\}$. We define a homotopy $H_\lambda: V \times [0, 1] \rightarrow U$ by the formula

$$H_\lambda(\{v_s\}, t) = \{H_s(v_s, t)\}, \text{ for any } v = \{v_s\} \in V,$$

where

$$H_s: V_s \times [0, 1] = \mathbb{T}_s \times [0, 1] \rightarrow \mathbb{T}_s = U_s$$

for each $s \notin \{s_1, s_2, \dots, s_n\}$ is a homotopy such that

$$H_s(v_s, 0) = v_s \text{ and } H_s(v_s, 1) = \lambda_s$$

for any $v_s \in V_s$ (for each $s \in S$ the space \mathbb{T}_s is contractible) **and the proof is complete.** \square

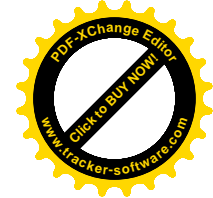
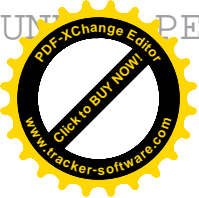
From the continuity of the Čech homology, we get:

Proposition 3.3. Let S be a nonempty set and let $X = \prod_{s \in S} X_s$, where for each $s \in S$ and the space X_s is compact. The space X is acyclic if and only if for each $s \in S$ a space X_s is acyclic. **Then, the space X is**

remove 'where'

Proof. Assume that X is acyclic and let $x \in X$, $x = \{x_s\}$ be an arbitrary point, $s_0 \in S$. **We show that a space X_{s_0} is acyclic.** **We have a diagram** Consider the map **We want to show that the**

$$X_{s_0} \xrightarrow{h} X \xrightarrow{\pi_{s_0}} X_{s_0},$$



where $h(z_{s_0}) = \{y_s\}$, $y_s = x_s$ for each $s \neq s_0$, $y_{s_0} = z_{s_0}$ and π_{s_0} is a projection. We observe that

$$\pi_{s_0} \circ h = Id_{X_{s_0}}.$$

Hence, the map

$$h_* : H_*(X_{s_0}) \rightarrow H_*(X)$$

is a monomorphism, so X_{s_0} is an acyclic space. Assume now that, for each $s \in S$ the space X_s is acyclic. From the mathematical literature (see, [5]) we know that if the spaces X_1, \dots, X_n are acyclic, then the space $X_1 \times \dots \times X_n$ is acyclic. Let $\Sigma = \{\xi \subset S : \xi \text{ is a finite set}\}$, then (Σ, \leq) is a directed set, where \leq is an inclusion. We have

Given that

$$X = \lim_{\leftarrow} \{Y_\xi, \pi_\xi^\zeta, \Sigma\},$$

where $Y_\xi = X_{s_1} \times X_{s_2} \times \dots \times X_{s_n}$, $\xi = \{s_1, s_2, \dots, s_n\} \subset S$ and for each $\zeta \leq \xi$, $\pi_\xi^\zeta : Y_\xi \rightarrow Y_\zeta$ is a projection. Hence and from the continuity of the Čech homology the space X is acyclic and the proof is complete. It follows from the....that the space X is acyclic. \square

Let Q be a Hilbert cube. In particular, we have:

Proposition 3.4. Let S be a nonempty set and let $X = \prod_{s \in S} X_s$. Assume that, for each $s \in S$, X_s is a compact subset of $Q_s = Q$. The space X has a trivial shape in $\mathbb{T} = \prod_{s \in S} Q_s$ if and only if for each $s \in S$ a space X_s has a trivial shape in Q_s .

Proposition 3.5. If $X \subset \mathbb{T}$ has a trivial shape in \mathbb{T} , then X is acyclic.

Proof. We denote by Σ Let

$$\Sigma = \{K_\xi : K \text{ is a compact neighborhood of } X \text{ in } \mathbb{T}\}$$

a directed set from the inclusion \leq , that is, $(\xi \leq \zeta) \Leftrightarrow (K_\zeta \subset K_\xi)$ for each $\xi, \zeta \in \Sigma$ and let

$$\mathbf{X} = \{K_\zeta, j_\xi^\zeta, \Sigma\}$$

be an inverse system, where for $\xi \leq \zeta$, $j_\xi^\zeta : K_\zeta \rightarrow K_\xi$ is an inclusion. We observe that from the assumption the inclusion $i_\xi : X \rightarrow K_\xi$ is homotopic to a constant map $C_\xi : X \rightarrow K_\xi$, $C_\xi(x) = x_0$ for each $x \in X$, for each $\xi \in \Sigma$, where $x_0 \in X$ is an arbitrary point. Hence and from the continuity of the Čech homology we have

Then,

Since Čech homology is continuous, we obtain $(\lim_{\leftarrow} i_\xi)_* = (\lim_{\leftarrow} C_\xi)_*$, and an arbitrary

where

$$(\lim_{\leftarrow} i_\xi)_* : H_*(X) \rightarrow H_*(\lim_{\leftarrow} \mathbf{X})$$

is an isomorphism, so X is acyclic and the proof is complete. \square

Reference all your propositions

What are you trying to do here?

Definition (Cell-like)

Let X be a space. A perfect map $\alpha : Z \rightarrow X$ is cell-like if for each compact set $K \subset X$ there exist a Tychonoff cube \mathbb{T} and an embedding $h : \alpha^{-1}(K) \rightarrow \mathbb{T}$ such that for each $x \in K$ the set $h(\alpha^{-1}(x))$ has a trivial shape in \mathbb{T} . Let Δ be family sets of compact and nonempty spaces such that the following conditions are satisfied:

(3) if A is a single-element space, then $A \in \Delta$.

(4) If, for each $s \in S$, $A_s \in \Delta$ then $\left(\prod_{s \in S} A_s\right) \in \Delta$,

where S is any, nonempty set. For each $A \in \Delta$ there exists a Tychonoff cube \mathbb{T} and an embedding $h : A \rightarrow \mathbb{T}$ such that

(5) $h(A) \in \Delta$.

If $A \in \Delta$ is a metrizable space then there exists an embedding $h : A \rightarrow Q$ such that

(6) $h(A) \in \Delta$

where Q is a Hilbert cube. Let X be a space. We will say that a perfect map $\alpha : Z \rightarrow X$ is a Δ map if for each compact set $K \subset X$ there exist a Tychonoff cube \mathbb{T} and an embedding $h : \alpha^{-1}(K) \rightarrow \mathbb{T}$ such that for each $x \in K$ the set $h(\alpha^{-1}(x)) \in \Delta$. We observe that if $\alpha \in \mathbb{D}(X)$, then for each nonempty set $B \subset X$ (not necessarily compact) $\alpha_{\alpha^{-1}(B)} \in \mathbb{D}(B)$, where $\alpha_{\alpha^{-1}(B)}$ is a restriction of α to the set $\alpha^{-1}(B)$. We will denote by

(7) $\mathbb{D}(X) = \{\alpha : Z \rightarrow X; \alpha \text{ is a } \Delta \text{ map}\}.$

~~The examples of families of \mathbb{D} type sets:~~ The following are examples of \mathbb{D} type sets;

(8) $\text{HOM}(X) = \{\alpha : Z \rightarrow X; \alpha \text{ is a homeomorphism}\},$

(9) $\text{CELL}(X) = \{\alpha : Z \rightarrow X; \alpha \text{ is a cell-like map}\},$

(10) $\mathbb{V}(X) = \{\alpha : Z \rightarrow X; \alpha \text{ is a Vietoris map}\}.$

We observe that

How? Put the reference

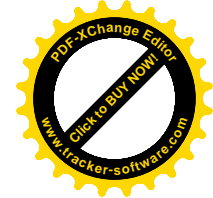
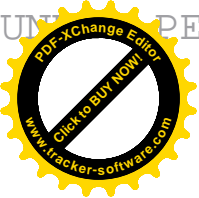
(11) $\text{HOM}(X) \subset \text{CELL}(X) \subset \mathbb{V}(X).$

4 Relative extensions of maps

section,

In this paragraph we will define the notion of the relative extension of maps and we will prove a few of their properties.

some of its



Definition 4.1. We say that a space X is a relative extension (relative neighborhood extension) (we write $X \in ESR(\mathbb{D})$, ($X \in NESR(\mathbb{D})$)) if for each compact set $K \subset X$ there exists a space Z_K , $\alpha_K : Z_K \rightarrow K$, $\alpha_K \in \mathbb{D}(K)$ such that for each compact space Y , for each closed set $A \subset Y$ and for each continuous map $f : A \rightarrow Z_K$ the map $\alpha_K \circ f$ has a continuous extension $F : Y \rightarrow X$ ($F : U \rightarrow X$), where $U \subset Y$ is some open set such that $A \subset U$, that is the following diagram:

$$\begin{array}{ccc} Z_K & \xrightarrow{j \circ \alpha_K} & X \\ \uparrow f & & \uparrow F \\ A & \xrightarrow{i} & T, \end{array}$$

is commutative, where $T = Y$ ($T = U$), $i : A \hookrightarrow T$ and $j : K \hookrightarrow X$ are inclusions.

There should be a statement before mathematical expression. Reconstruct Pro. 4.2
Proposition 4.2. $X \in NESR(\mathbb{HOM})(ESR(\mathbb{HOM})) \Leftrightarrow X \in NES(ES)$.

Proof. Let $X \in NESR(\mathbb{HOM})$ and let $f : A \rightarrow X$ be a continuous map, where $A \subset Y$ is a closed subset of a compact space Y . We denote by $K = f(A)$. There exists a map $\alpha_K : Z_K \rightarrow K$, $\alpha_K \in \mathbb{HOM}(K)$ such that the conditions of Definition 4.1 are satisfied.

We have a diagram

Given that

$$A \xrightarrow{\tilde{f}} K \xrightarrow{\alpha_K^{-1}} Z_K \xrightarrow{\alpha_K} K \xrightarrow{i} X,$$

where $\tilde{f}(y) = f(y)$ for each $y \in A$, α_K^{-1} is an inverse homeomorphism and i is an inclusion. There exists an extension $G : U \rightarrow X$ of $\alpha_K \circ (\alpha_K^{-1} \circ \tilde{f}) = \tilde{f}$, where $U \subset Y$ is some open neighborhood of A . The proof in the opposite direction is obvious and the proof of the second part of this Proposition is analogical. \square

Let X, Y be metrizable spaces. We recall that a space X is a \mathbb{D} -retract of Y , if there exist a metrizable space $Z \subset Y$, $\alpha : Z \rightarrow X$, $\alpha \in \mathbb{D}(X)$ and $r : Y \rightarrow X$ such that $r \circ i = \alpha$, where $i : Z \hookrightarrow Y$ is an inclusion. We will say that the map r is a \mathbb{D} -retraction and a space Z is a \mathbb{D} -carrier of X in Y . We will write $X \in ANRR(\mathbb{D})$ ($X \in ARR(\mathbb{D})$) if there exists a normed space E , an open set $V \subset E$ such that X is a \mathbb{D} -retract of V (E) (see, [9]). We observe that if $\alpha : X \rightarrow Y$ is a perfect (proper) map such that, for each $y \in Y$, the set $\alpha^{-1}(y)$ has a trivial shape then $\alpha \in \mathbb{CELL}(Y)$ (see (6) and Proposition 3.4). It is obvious that if $\alpha : X \rightarrow Y$ is a homeomorphism then $\alpha \in \mathbb{HOM}(Y)$.

Try and explain what you are trying to do here.

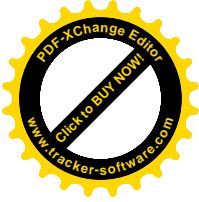
Proposition 4.3. Let X be a metrizable space. If

$$(X \in ANRR(\mathbb{D})(ARR(\mathbb{D}))) \xRightarrow{\text{then}} (X \in NESR(\mathbb{D})(ESR(\mathbb{D}))).$$

Then, there exists

Proof. Let X be a metrizable space and let $X \in ANRR(\mathbb{D})$. **There exist** a normed space E , an open set $V \subset E$ such that X is a \mathbb{D} -retract of V . That is, there exist a space $Z \subset V$, $\alpha : Z \rightarrow X$, $\alpha \in \mathbb{D}(X)$ and $r : V \rightarrow X$ such that $r \circ i = \alpha$, where $i : Z \hookrightarrow V$ is an inclusion. Let $K \subset X$ be a compact set, $Z_K = \alpha^{-1}(K)$ and $\alpha_K : Z_K \rightarrow K$, $\alpha_K(z) = \alpha(z)$ for each $z \in Z_K$, where $\alpha_K \in \mathbb{D}(K)$. **Let's take** a compact space Y , a closed set $A \subset Y$ and a continuous map $f : A \rightarrow Z_K$. **We have a following diagram:**

$$A \xrightarrow{f} Z_K \xrightarrow{j} Z \xrightarrow{i} V \xrightarrow{r} X,$$



where j is an inclusion. There exists an extension $G : U \rightarrow V$ of $i \circ j \circ f$ (see, Remark 2.6), where $U \subset Y$ is some open neighborhood of A . We define an extension $F : U \rightarrow X$ of $\alpha_K \circ f$ by the formula

$$F = r \circ G.$$

The proof of the second part of this Proposition is analogical. \square

Let X, Y, X_i be compact spaces, $i = 1, 2$. It's clear that if $\alpha \in \mathbb{D}(X)$ and $h \in \text{HOM}(Y)$, $h : X \rightarrow Y$, then $(h \circ \alpha) \in \mathbb{D}(Y)$. We observe also that if $\alpha_1 \in \mathbb{D}(X_1)$ and $\alpha_2 \in \mathbb{D}(X_2)$ then $(\alpha_1 \times \alpha_2) \in \mathbb{D}(X_1 \times X_2)$.

Proposition 4.4. $((X_1 \times X_2) \in \text{NESR}(\mathbb{D})(\text{ESR}(\mathbb{D})) \Leftrightarrow (X_1 \in \text{NESR}(\mathbb{D})(\text{ESR}(\mathbb{D}))$
and $X_2 \in \text{NESR}(\mathbb{D})(\text{ESR}(\mathbb{D})))$.

Proof. Let $(X_1 \times X_2) \in \text{NESR}(\mathbb{D})$ and let $K \subset X_1 (K \subset X_2)$ be a compact set. Then a set $K \times \{x_2\} \subset X_1 \times X_2$ ($\{x_1\} \times K \subset X_1 \times X_2$) is compact, where $(x_1, x_2) \in X_1 \times X_2$ is an arbitrary point. From the assumption there exists a map $\alpha_K : Z_K \rightarrow K \times \{x_2\}$ ($\alpha_K : Z_K \rightarrow \{x_1\} \times K$), $\alpha_K \in \mathbb{D}(K \times \{x_2\})$ ($\alpha_K \in \mathbb{D}(\{x_1\} \times K)$) such that the conditions of Definition 4.1 are satisfied. Let T be a compact space and let $f : A \rightarrow Z_K$ be a continuous map, where $A \subset T$ is a closed set. There exists an extension $G : U \rightarrow X_1 \times X_2$ of $\alpha_K \circ f$, where $U \subset T$ is an open neighborhood of A . We define a map $F : U \rightarrow X_1$ ($F : U \rightarrow X_2$) by the formula

$$F = \pi_1 \circ G \quad (F = \pi_2 \circ G),$$

where $\pi_i : X_1 \times X_2 \rightarrow X_i$ are projections, $i = 1, 2$. Now, let $X_1 \in \text{NESR}(\mathbb{D})$ and $X_2 \in \text{NESR}(\mathbb{D})$ and let $K \subset X_1 \times X_2$ be a compact set. We denote by $K_i = \pi_i(K)$, where $\pi_i : X_1 \times X_2 \rightarrow X_i$ are projections, $i = 1, 2$. There exists a map $\alpha_{K_i} : Z_{K_i} \rightarrow K_i$, $\alpha_{K_i} \in \mathbb{D}(K_i)$, $i = 1, 2$ such that the conditions of Definition 4.1 are satisfied. It's obvious that $K \subset K_1 \times K_2$. Let $\alpha = \alpha_{K_1} \times \alpha_{K_2}$, $Z_K = \alpha^{-1}(K)$ and $\alpha_K : Z_K \rightarrow K$, $\alpha_K(z) = \alpha(z)$ for each $z \in Z_K$. Let T be a compact space and let $f : A \rightarrow Z_K$ be a continuous map, where $A \subset T$ is a closed set. For $i = 1, 2$ we have

$$A \xrightarrow{f} Z_K \xrightarrow{j} Z_{K_1} \times Z_{K_2} \xrightarrow{\pi'_i} Z_{K_i} \xrightarrow{\alpha_{K_i}} K_i \xrightarrow{j_i} X_i,$$

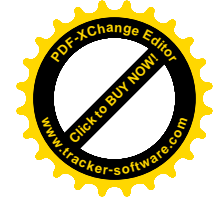
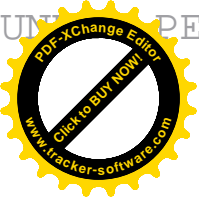
where j, j_i are inclusions and $\pi'_i : Z_{K_1} \times Z_{K_2} \rightarrow Z_{K_i}$ are projections, $i = 1, 2$. There exist extensions $G_i : V_i \rightarrow X_i$ of $\alpha_{K_i} \circ (\pi'_i \circ j \circ f)$, where $V_i \subset T$ some open neighborhoods of A , $i = 1, 2$. Let $U = V_1 \cap V_2$. We define an extension $F : U \rightarrow X_1 \times X_2$ of $\alpha_K \circ f$ by the formula

$$F(t) = (G_1(t), G_2(t)) \quad \text{for each } t \in U.$$

The proof of the second part of this Proposition is analogical. \square

Proposition 4.5. Let S be a nonempty set and let $X = \prod_{s \in S} X_s$. If $X \in \text{NESR}(\mathbb{D})$ then $X_s \in \text{NESR}(\mathbb{D})$ for each $s \in S$.

Proof. Assume that $X \in \text{NESR}(\mathbb{D})$. Let $s_0 \in S$, $K \subset X_{s_0}$ be a compact set and let $\{x_s\} \in X$ be an arbitrary point. For $P = \prod_{s \in S} Y_s$, where for each $s \neq s_0$, $Y_s = \{x_s\}$ and $Y_{s_0} = K$, there exists a map $\alpha_P \in \mathbb{D}(K)$, $\alpha_P : Z_P \rightarrow P$ such that the conditions



of Definition 4.1 are satisfied. Let $Z_K = Z_P$ and let $\alpha_K = h \circ \alpha_P$, where $h : P \rightarrow K$ is a homeomorphism (restriction of a projection $\pi_{s_0} : X \rightarrow X_{s_0}$). Let $f : A \rightarrow Z_K$ be a continuous map, where A is a closed subset of a compact space Y . There exists an extension $G : U \rightarrow X$ of $\alpha_K \circ f$, where $U \subset Y$ is an open neighborhood of A . We define an extension $F : U \rightarrow X_{s_0}$ of $\alpha_K \circ f$ by the formula $F = \pi_{s_0} \circ G$ and the proof is complete. \square

Proposition 4.6. *Let S be a nonempty set and let $X = \prod_{s \in S} X_s$. A space $X \in ESR(\mathbb{D})$ if and only if $X_s \in ESR(\mathbb{D})$ for each $s \in S$.*

Proof. Assume that $X \in ESR(\mathbb{D})$. Then from the proof of Proposition 4.5 it results that $X_s \in ESR(\mathbb{D})$ for each $s \in S$. Let for each $s \in S$ the space $X_s \in ESR(\mathbb{D})$ and let $K \subset X$ be a compact set. We denote by $K_s = \pi_s(K)$, where $\pi_s : X \rightarrow X_s$ is a projection for each $s \in S$. For each $s \in S$ we take a map $\alpha_{K_s} \in \mathbb{D}(K_s)$, $\alpha_{K_s} : Z_{K_s} \rightarrow K_s$ such that the conditions of Definition 4.1 are satisfied. It's obvious that $K \subset \prod_{s \in S} K_s$. Let $\alpha = \prod_{s \in S} \alpha_{K_s} : \prod_{s \in S} Z_{K_s} \rightarrow \prod_{s \in S} K_s$, $Z_K = \alpha^{-1}(K)$ and let $\alpha_K : Z_K \rightarrow K$ be a restriction of α . It's clear that $\alpha_K \in \mathbb{D}(K)$. Let $f : A \rightarrow Z_K$ be a map, where A is a closed subset of a compact space Y and let $\pi_s^1 : Z_K \rightarrow Z_{K_s}$ be a projection for each $s \in S$. We have a following diagram:

$$A \xrightarrow{f} Z_K \xrightarrow{\pi_s^1} Z_{K_s} \xrightarrow{\alpha_{K_s}} K_s \xrightarrow{i_s} X_s,$$

where i_s is an inclusion, for each $s \in S$. From the assumption, for each $s \in S$, there exists an extension $F_s : Y \rightarrow X_s$ of $\alpha_{K_s} \circ (\pi_s^1 \circ f)$. We define an extension $F : Y \rightarrow X$ of $\alpha_K \circ f$ by the formula:

$$F(y) = \{F_s(y)\} \quad \text{for each } y \in Y.$$

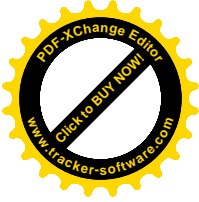
and the proof is complete. \square

Example 4.7. *Let $p : Q \rightarrow X$ be a cell-like map such that X is a metrizable and non-movable space (see, [7]), where Q is a Hilbert cube. Let S be a nonempty set ($\text{card}(S) > \aleph_0$) and let $Y = \prod_{s \in S} Y_s$, where $Y_s = X$ for each $s \in S$. We define a cell-like map $\alpha : \mathbb{T} \rightarrow Y$ by the formula $\alpha = \{p_s\}_{s \in S}$, where $\mathbb{T} = \prod_{s \in S} Q_s$ is a Tychonoff cube and for each $s \in S$, $Q_s = Q$ and $p_s = p$. It's clear that $Y \in ESR(\text{CELL})$ (in particular, $Y \in NESR(\text{CELL})$) is a non-metrizable space. We show that $Y \notin NES$. Assume that $Y \in NES$. Then, from Proposition 4.5, $X \in NES$. Hence X is a neighborhood retract of Q , so $X \in ANR$, but it is a contradiction, since X is a non-movable space (see Remark 2.9).*

5 The abstract morphism

The symbol $D(X, Y)$ will denote the set of all diagrams of the form

$$X \xleftarrow{p} Z \xrightarrow{q} Y,$$



where $p : Z \rightarrow X$ denotes a Vietoris map and $q : Z \rightarrow Y$ denotes a continuous map. Each such diagram will be denoted by (p, q) . Let $(p_1, q_1) \in D(X, Y)$ and $(p_2, q_2) \in D(Y, T)$. The composition of diagrams (see, [5])

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y \xleftarrow{p_2} Z_2 \xrightarrow{q_2} T,$$

is called a diagram $(p, q) \in D(X, T)$

$$X \xleftarrow{p} Z_1 \triangle_{q_1 p_2} Z_2 \xrightarrow{q} T,$$

$$\text{where } Z_1 \triangle_{q_1 p_2} Z_2 = \{(z_1, z_2) \in Z_1 \times Z_2 : q_1(z_1) = p_2(z_2)\},$$

$$p = p_1 \circ \pi_1, \quad q = q_2 \circ \pi_2,$$

$$Z_1 \xleftarrow{\pi_1} Z_1 \triangle_{q_1 p_2} Z_2 \xrightarrow{\pi_2} Z_2,$$

$$\pi_1(z_1, z_2) = z_1 \text{ (Vietoris map)}, \quad \pi_2(z_1, z_2) = z_2 \text{ for each } (z_1, z_2) \in Z.$$

It shall be written

$$(p, q) = (p_2, q_2) \circ (p_1, q_1).$$

From ([5], p. 201, 202) it also results that the composition of the diagrams satisfies the condition:

$$(12) \quad \text{for each } x \in X \quad q(p^{-1}(x)) = q_2(p_2^{-1}(q_1(p_1^{-1}(x)))).$$

Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$. Assume that in the set $D(X, Y)$ we have an equivalency relation (it's) denoted as \sim_a such that the following conditions are satisfied (see [14, 13, 8]):

$$(13) \quad ((p_1, q_1) \sim_a (p_2, q_2)) \Rightarrow (\text{for each } x \in X \quad q_1(p_1^{-1}(x)) = q_2(p_2^{-1}(x))),$$

$$(14) \quad ((p_1, q_1) \sim_a (p_2, q_2)) \Rightarrow (q_{1*} \circ p_{1*}^{-1} = q_{2*} \circ p_{2*}^{-1}),$$

Let $(p_3, q_3), (p_4, q_4) \in D(Y, T)$.

$$(15) \quad ((p_1, q_1) \sim_a (p_2, q_2) \text{ and } (p_3, q_3) \sim_a (p_4, q_4)) \Rightarrow (((p_3, q_3) \circ (p_1, q_1)) \sim_a ((p_4, q_4) \circ (p_2, q_2))).$$

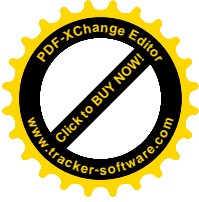
The set $M_a(X, Y) = D(X, Y) / \sim_a$ will be called a set of abstract morphisms (a -morphism). Let $(p, q) \in D(X, Y)$. For any $\varphi_a \in M_a(X, Y)$ the set $\varphi(x) = q(p^{-1}(x))$ where $\varphi_a = [(p, q)]_a$ is called an image of the point x in the a -morphism φ_a . We denote by $\varphi : X \rightarrow_a Y$ a multivalued map determined by $\varphi_a \in M_a(X, Y)$. We observe that from (12) and (15) it results that if $\varphi : X \rightarrow_a Y$ is determined by $\varphi_a = [(p_1, q_1)]_a$ and $\psi : Y \rightarrow_a T$ is determined by $\psi_a = [(p_2, q_2)]_a$, then $\psi \circ \varphi : X \rightarrow_a T$ is determined by

$$(\psi \circ \varphi)_a = [((p_2, q_2) \circ (p_1, q_1))]_a.$$

We recall that a multivalued map $\varphi : X \multimap Y$ is acyclic, if for each $x \in X$ the set $\varphi(x)$ is compact and acyclic. An acyclic map $\varphi : X \rightarrow_a Y$ because it's determined by $\varphi_a = [(p_\varphi, q_\varphi)]_a$, where

$$X \xleftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} Y$$

are maps given by formulas: $p_\varphi(x, y) = x$, $q_\varphi(x, y) = y$ for each $(x, y) \in \Gamma_\varphi$ and Γ_φ is a graph of φ .



Remark 5.1. Let $\varphi : X \rightarrow_a X$ be a map. We observe that a map φ has a fixed point i.e., there exists a point $x \in X$ such that $x \in \varphi(x)$ if and only if for some $(p, q) \in \varphi_a$, p and q have a coincidence point.

Let **TOP** denote categories in which Hausdorff topological spaces are objects and continuous mappings are category mappings. Let **TOP_a** denote categories in which Hausdorff topological spaces are objects and multivalued maps determined by abstract morphisms are category mappings. Let **VECT_G** denote categories in which linear graded vector spaces are objects and linear mappings of degree zero are category mappings.

Theorem 5.2. (see [14]) The mapping $\widetilde{\mathbf{H}}_* : \mathbf{TOP}_a \rightarrow \mathbf{VECT}_G$ given by the formula

$$\widetilde{\mathbf{H}}_*(\varphi) \equiv \varphi_* = q_* \circ p_*^{-1},$$

where φ is a multivalued map determined by $\varphi_a = [(p, q)]_a$ is a covariant functor and the extension of the functor of the Čech homology $\mathbf{H}_* : \mathbf{TOP} \rightarrow \mathbf{VECT}_G$.

Let $\varphi : X \rightarrow_a X$ be a map. Assume that φ_* is a Leray endomorphism. Then we define a Lefschetz number of φ_* by the formula

$$\Lambda(\varphi_*) = \Lambda(q_* \circ p_*^{-1}),$$

where (p, q) is some diagram in φ_a (see (14)). We recall that $\varphi : X \rightarrow_a X$ is a Lefschetz map if φ_* is a Leray endomorphism and $\Lambda(\varphi_*) \neq 0$ implies that the map φ has a fixed point.

Remark 5.3. A map $\varphi : X \rightarrow_a Y$ is admissible if and only if there exists a map $\Delta : X \rightarrow_a Y$ such that $\Delta \subset \varphi$, that is, for each $x \in X$, $\Delta(x) \subset \varphi(x)$.

6 The fixed points of compact maps

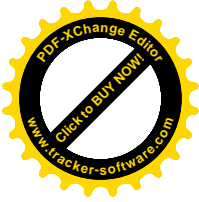
In this paragraph, we will show that the spaces of $NESR(\mathbb{V})$ type have the fixed point property.

Theorem 6.1. Let $X \in NESR(\mathbb{V})$ and let $\varphi : X \rightarrow_a X$ be a compact map. Then φ is a Lefschetz map.

Proof. For $K = \overline{\varphi(X)}$ there exists a map $\alpha_K : Z_K \rightarrow K$, $\alpha_K \in \mathbb{V}(K)$ such that the conditions of Definition 4.1 are satisfied. Let $h : Z_K \rightarrow \mathbb{T}$ be an embedding and let $S = h(Z_K) \subset \mathbb{T}$, where \mathbb{T} is some Tychonoff cube. We have the following diagrams:

$$\begin{array}{ccccc} S & \xrightarrow{h^{-1}} & Z_K & \xrightarrow{\alpha_K} & K & \xrightarrow{i_1} & X, \\ X & \xrightarrow{\tilde{\varphi}} & K & \xrightarrow{\overleftarrow{\alpha}_K} & Z_K & \xrightarrow{h} & S, \end{array}$$

where i_1 is an inclusion, h^{-1} is an inverse homeomorphism, $\tilde{\varphi}(x) = \varphi(x)$ for each $x \in X$ and $\overleftarrow{\alpha}_K(x) = \alpha_K^{-1}(x)$ for each $x \in K$. From the assumption there exists an extension $F : U \rightarrow X$ of $\alpha_K \circ h^{-1}$, where $U \subset \mathbb{T}$ is some open set such that $S \subset U$. We get the following commutative diagram:



$$\begin{array}{ccc} X & \xrightarrow{\psi} & U \\ \varphi \uparrow & \searrow F & \uparrow \psi \circ F \\ X & \xrightarrow{\psi} & U, \end{array}$$

where $\psi = i_2 \circ h \circ \overleftarrow{\alpha}_K \circ \tilde{\varphi}$ and $i_2 : S \hookrightarrow U$ is an inclusion. There exists a locally convex space $L(\mathbb{T})$ such that \mathbb{T} is a retract of $L(\mathbb{T})$ (see, [4]). Let $r : L(\mathbb{T}) \rightarrow \mathbb{T}$ be a retraction and let $\tilde{r} : r^{-1}(U) \rightarrow U$ be a map given by $\tilde{r}(x) = r(x)$ for each $x \in r^{-1}(U)$. We have the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{j} & r^{-1}(U) \\ \psi \circ F \uparrow & \searrow \eta & \uparrow j \circ \eta \\ U & \xrightarrow{j} & r^{-1}(U), \end{array}$$

where j is an inclusion and $\eta = \psi \circ F \circ \tilde{r}$. From Proposition 2.1 and Theorem 2.4 a Lefschetz number $\Lambda(\varphi_*)$ is well defined and $\Lambda(\varphi_*) = \Lambda((j \circ \eta)_*)$. Assume now, that $\Lambda(\varphi_*) \neq 0$, then from Theorem 2.4 the map $j \circ \eta$ has a fixed point (see, Remark 5.1). Hence, it results that a map $\psi \circ F$ has a fixed point. Let $x \in U$ be a fixed point of $\psi \circ F$. We have

$$F(x) \in F(h(\overleftarrow{\alpha}_K(\tilde{\varphi}(F(x)))))) = \tilde{\varphi}(F(x)) = \varphi(F(x)).$$

Thus, φ is a Lefschetz map and the proof is complete. \square

7 The fixed points of noncompact maps

We will use the following denotation $\varphi^n \equiv \varphi \circ \varphi \circ \dots \circ \varphi$, (n th iterate of φ), where $n \in \mathbb{N}$.

Definition 7.1. A map $\varphi : X \rightarrow_a X$ is called a compact absorbing contraction (written $\varphi \in CAC(X)$) provided there exists an open set $U \subset X$ such that:

(7.1.1) $\varphi(U) \subset U$ and the map $\varphi_U : U \rightarrow_a U$, $\varphi_U(x) = \varphi(x)$ for every $x \in X$ is compact,

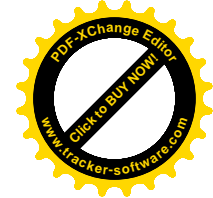
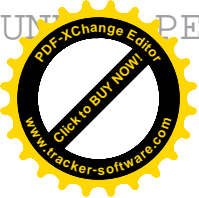
(7.1.2) for every $x \in X$ there exists $n = n_x$ such that $\varphi^n(x) \subset U$.

Proposition 7.2. (see [5]) Let $\varphi \in CAC(X)$ and U be an open subset X as in Definition 7.1. If K is a compact subset of X , then there exists $n \in \mathbb{N}$ such that $\varphi^n(K) \subset U$.

Let $\varphi : X \rightarrow Y$ be a map and let $A \subset X$ and $B \subset Y$ be nonempty sets. Assume that $\varphi(A) \subset B$. We denote by $\hat{\varphi} : (X, A) \rightarrow (Y, B)$ a map of pairs, that is, $\hat{\varphi}(x) = \varphi(x)$ for each $x \in X$.

Proposition 7.3. (see [5]) Let $\hat{\varphi} : (X, A) \rightarrow_a (X, A)$ be a map of pairs. If any two of endomorphisms $\hat{\varphi}_* : H(X, A) \rightarrow H(X, A)$, $\varphi_* : H(X) \rightarrow H(X)$, $\varphi_{A*} : H(A) \rightarrow H(A)$ are Leray endomorphisms, then so is the third and

$$\Lambda(\hat{\varphi}_*) = \Lambda(\varphi_*) - \Lambda(\varphi_{A*}).$$



Proposition 7.4. *Let $X \in NESR(\mathbb{D})$. If $U \subset X$ is an open set then $U \in NESR(\mathbb{D})$.*

Proof. Let $U \subset X$ be an open set and let $K \subset U$ be a compact set. There exists $\alpha_K : Z_K \rightarrow K$, $\alpha_K \in \mathbb{D}(K)$ such that the conditions of Definition 4.1 are satisfied. Let Y be a compact space and let $f : A \rightarrow Z_K$ be a continuous map, where $A \subset Y$ is a closed set. From the assumption there exists an extension $F_1 : V_1 \rightarrow X$ of $\alpha_K \circ f$, where $V_1 \subset Y$ is some open neighborhood of A . The set $V = F_1^{-1}(U) \subset Y$ is an open neighborhood of A . We define an extension $F : V \rightarrow U$ of $\alpha_K \circ f$ given by the formula

$$F(x) = F_1(x) \text{ for each } x \in V$$

and the proof is complete. \square

Theorem 7.5. *Let X be a space and let $\varphi \in CAC(X)$. Assume further that there exists a space $A \subset X$ such that $A \in NESR(\mathbb{V})$ and $\overline{\varphi(U)} \subset A$, where U is chosen according to Definition 7.1, then φ is a Lefschetz map.*

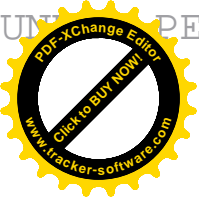
Proof. Let $\psi : U \rightarrow_a U \cap A$ be a map given by $\psi(x) = \varphi(x)$ for all $x \in U$. By the assumption, a map ψ is well-defined. We observe that $(U \cap A) \in NESR(\mathbb{V})$ (see, Proposition 7.4). A homomorphism $\hat{\varphi}_* : H(X, U) \rightarrow H(X, U)$ is weakly nilpotent (see, [12]). Hence, and from Proposition 2.2 we get $\Lambda(\hat{\varphi}_*) = 0$. We have a following commutative diagram:

$$\begin{array}{ccc} H(U \cap A) & \xrightarrow{i_*} & H(U) \\ \varphi_{A \cap U*} \uparrow & \searrow \psi_* & \uparrow \varphi_{U*} \\ H(U \cap A) & \xrightarrow{i_*} & H(U), \end{array}$$

where $i : U \cap A \hookrightarrow U$ is an inclusion. From the above diagram and Proposition 2.1, it results that φ_{U*} is a Leray endomorphism and $\Lambda(\varphi_{U*}) = \Lambda(\varphi_{A \cap U*})$ (see, Theorem 6.1). Hence, and from Proposition 7.3, we get that φ_* is a Leray endomorphism and $\Lambda(\varphi_*) = \Lambda(\varphi_{U*})$. Assume that $\Lambda(\varphi_*) \neq 0$. Then $\Lambda(\varphi_{A \cap U*}) \neq 0$ and $\varphi_{A \cap U}$ has a fixed point (see, Theorem 6.1). It's clear that $\text{Fix}(\varphi_{A \cap U}) \subset \text{Fix}(\varphi)$, so φ is a Lefschetz map and the proof is complete. \square

8 Conclusion

In paragraph 3 the notions of a trivial shape in topological spaces are given. In paragraph 4 the notions of ES and NES are generalized. The Example 4.7 shows that the class of spaces of $NESR(\text{CELL})$ ($ESR(\text{CELL})$) type is essentially wider than the class of spaces of NES (ES) type. We prove that in the class of metrizable spaces $ANRR(\mathbb{D}) \subset NESR(\mathbb{D})$ ($ARR(\mathbb{D}) \subset ESR(\mathbb{D})$). In paragraphs 6 and 7, we prove that the spaces of $NESR(\mathbb{V})$ type (in particular $NESR(\text{CELL})$) have the fixed point property (see Theorem 6.1 and Theorem 7.5). It's worth mentioning that this article is strongly related to [10, 9, 11].



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