Large deviation for stochastic differential systems pertubated by a rapid process in the Besov-Orlicz topology

Abstract

In this paper, we study a large deviations principle associated a familly process X^ε perturbated by a rapid process ζ in the Besov-Orlicz space. The process X^ε is a solution of Itô integral :

$$\begin{cases} dX_t^\varepsilon &= b(X_t^\varepsilon, \zeta_{t/\varepsilon})\, dt + \sqrt{\varepsilon}\, \sigma(X_t^\varepsilon)\, dW_t \\ X_0 &= x \in \mathbb{R}^d \end{cases}$$

with the condition ζ is independant of the brownian motion W and obeys a large deviation principle.
Key words :Large deviations, averaging principle, Besov-Orlicz space

1 Introduction

In this paper, we consider a family processes X^ϵ d-dimensional solution of stochastic differential equations :

$$dX_t^\varepsilon = b(X_t^\epsilon, \zeta_{t/\varepsilon})\ dt + \sqrt{\varepsilon}\ \sigma(X_t^\varepsilon)\ dW_t, \quad X_0 = x \in \mathbb{R}^d \tag{1}$$

where W is a standard Wiener process independant of $\zeta_{t/\varepsilon}$. This is to obtain the asymptotic evaluation of $P(X_t^\epsilon \in A)$ where A is a Borel set of Besov-Orlicz space under the assumptions that the process X_t^ϵ converges to the solution $\bar{X}_t$ defined by :

$$\begin{aligned} d\bar{X}_t &= \bar{b}(\bar{X}_t)\ \ dt, \quad \bar{X}_0 = 0, \\ \bar{b}(x) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T b(x_s, \zeta_{s/\epsilon})\ ds \end{aligned} \tag{2}$$

The asymptotic evaluation obtained will be a result of large deviations from X_t^ϵ compared to $\bar{X}_t$.

The basic work on the subject is the article by Freidlin [10], see also Ventcel's book - Freidlin [9]where he gets this evaluation under the assumption :

$$\lim_{T\to\infty} \frac{1}{T} \log E\left(\exp(\int_0^T \left\langle \alpha, b(x_s, \zeta_{s/\epsilon})\right\rangle)\ ds\right) = H^0(x, \alpha) \tag{3}$$

exists uniformly in x and differentiable in α.

The special case $\zeta \equiv 0\ \ (b(X_t^\epsilon, 0) = b(X_t^\epsilon))$, $\sigma \not\equiv Id$ was studied by Freidlin & Wentzell [8] see also Varadhan [27], Azencott [1] and Stroock [26] with the usual topology of uniform convergence, Ben Arous and Ledoux [5] have been developed a large deviations principle(LDP) in Hölder spaces. Later on, an extension to Besov spaces was considered in Eddahbi et al [7] and Roynette [4]

The case $\zeta \not\equiv 0$ was studied by A. GUILLIN [13] in a situation of moderate deviations.

The aim of this paper is to study the large deviation principle (LDP) of the law of $\{X_t^\epsilon, \epsilon > 0\}$ in the Besov-Orlicz topology, that is we want to generalize the result of H. LAPEYRE [17] in the usual uniform topology.

The paper is organized as follows. In section 2, we introduce some hypotheses and notations. Section 3 contains some preliminaries definitions and results which are essential for the proof of the theorem. Section 4, under the hypotheses in section 2, we prove in theorem (4.3) the LDP of X_t^ϵ, solution of (1) when ζ satisfy a large deviations principle.

2 Hypotheses and Notations

2.1 Hypotheses

In this paper, we assume that the following hypotheses will be verified :

H1. The function $\sigma : \mathbb{R}^l \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^r$ is jointly measurable in (x, y) and there exists a constant $C > 0$ such that.

$$\begin{aligned} &|\sigma(x,y) - \sigma(x',y')| \leqslant C(|y-y'| + |x-x'|) \\ &|\sigma(x,y)| \leqslant C \end{aligned}$$

H2. The function $b : \mathbb{R}^l \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is jointly measurable in (x, y) and there exists a constant $C > 0$ such that.

$$\begin{aligned} &|b(x,y) - b(x',y')| \leqslant C(|y-y'| + |x-x'|) \\ &|b(x,y)| \leqslant C|x-y| \end{aligned}$$

H3. W is a standard $\mathbb{R}^r$-valued Brownian motion

H4. $\zeta_{t/\epsilon}$ is a process R^l-value independant of brownian motion W and obeys a large deviation principle with a good rate function I.

2.2 Notations

2.2.1 Cameron-Martin space

Let $H(\mathbb{R}^d)$ be the Cameron-Martin space associated with the Brownian motion on $\mathbb{R}^d$

$$H(\mathbb{R}^d) = \left\{ \begin{array}{c} f : [0,1] \to \mathbb{R}^d, f \text{ is absolutely continuous such that} \\ f(0) = 0 \text{ et } \int_0^1 |\dot{f}_s|^2 \, ds < +\infty \end{array} \right\}$$

$H(\mathbb{R}^d)$ is a Hilbert Space equipped with the norm

$$\langle f, g \rangle = \int_0^1 |\dot{f}_s|^2 |\dot{g}_s|^2 \, ds$$

2.2.2 Besov-Orlicz space

Let B_{M_β, w_α} be denote the Besov-Orlicz space of continuous function $f : [0,1] \to \mathbb{R}^d$ such that $\| f \|_{M_\beta, w_\alpha} < \infty$. For all $\alpha > 0$, Let us put

$$\| f \|_{M_\beta, w_\alpha,} = \| f \|_{M_\beta} + \sup_{0 \leq t \leq 1} \frac{w_{M_\beta}(f,t)}{w_{\alpha,\lambda}(t)}$$

where $w_{\alpha,\lambda}(t) = t^{\alpha}\left(1+\log\frac{1}{t}\right)^{\lambda}, \forall\alpha > 0, \| f \|_{M_\beta} = \inf\left\{\tau > 0, \frac{1}{\tau}\left[1+\int_0^1 M_\beta(\tau|f(t)|)dt\right]\right\}$ et $w_{M_\beta}(f,t) = \sup_{0\le h\le t} \| \Delta_h f \|_{M_\beta}$ with

$$\Delta_h f(x) = 1_{[0,1-h]}(x)(f(x+h) - f(x)), \forall h \in [0,1].$$

We will use the equivalent of Cieleski, Z. [4]. Let $\chi_1, \chi_{j,k}, j = 0, 1..., k = 1...2^j, supp\chi_{j,k} = [(k-1)/2^j, k/2^j]$, be the set of Haar functions over the interval $[0,1]$, and let $\varphi_0(t) = 1, \varphi_1(t) = t, \varphi_{j,k}(t) = \int_0^t \chi_{j,k}(s)ds$ be the set of Schauder functions. For all continuous functions $f : [0,1] \to \mathbb{R}^d$, soit $\{A_n(f), n \ge 0\}$ its development in series of Schauder is given by

$$f(t) = A_0(f)\varphi_0(t) + A_1(f)\varphi_1(t) + \sum_{n=2^j+1}^{2^{j+1}} \sum_{j,k} A_n(f)\varphi_{j,k}(t)$$

où $A_0(f) = f(0), A_1(f) = f(1) - f(0)$ and

$$A_n(f) = 2^{\frac{j}{2}}\left[\left(f\left(\frac{2k-1}{2^{j+1}}\right) - f\left(\frac{2k-2}{2^{j+1}}\right)\right) - \left(f\left(\frac{2k}{2^{j+1}}\right) - f\left(\frac{2k-1}{2^{j+1}}\right)\right)\right]$$

sont les coefficients de f dans cette base.

Let $B^0_{M_\beta,w_\alpha}$ be the subspace of B_{M_β,w_α} corresponding to the sequences $f_{j,k}$ such that

$$B^0_{M_\beta,w_\alpha} = \left\{f \in \mathcal{C}([0,1],\mathbb{R}^d); \| f \|_{M_2,w_\alpha} < \infty, \lim_{j\wedge p\to\infty} 2^{-j(\frac{1}{2}-\alpha+\frac{1}{p})}p^{-\gamma}(1+j)^{-\lambda} \| f_{j,.} \|_p = 0\right\}$$

where

$$\| f_{j,.} \|_p = \left(\sum_{k=1}^{2^j} |f_{j,k}|^p\right)^{\frac{1}{p}} \ et\ \beta\gamma = 1$$

$B^0_{M_\beta,w_\alpha}$ is a Banach space.

3 Preliminaries definitions and results

3.1 Preliminaries definitions

Definition 3.1. A function $I : E \longrightarrow [0; +\infty]$ is said to be a rate function if it is lower semicontinuous (lsc).
Furthermore, if for each $a < +\infty$, $\Gamma_a = \{x \in E, I(x) \leq a\}$ is compact, we will say that I is a good rate function.
Unless explicitly stated otherwise, for any subset A of E and any rate function, we set $I(A) = \inf_{x \in A} I(x)$.

Definition 3.2. For some function I, the family of probabilities $\{P^\varepsilon\}_{\varepsilon>0}$ satisfy a large-deviation principle if the following hold :

i) (Lower bound.) For every open subset O of E

$$\liminf_{\varepsilon \longrightarrow 0} \varepsilon \log P_\varepsilon(O) \geq -I(O)$$

ii) (Upper bound.) For every closed subset F of E

$$\limsup_{\varepsilon \longrightarrow 0} \varepsilon \log P_\varepsilon(F) \leq -I(F).$$

3.2 Preliminaries results

Theorem 3.3. *Let $p_0 \geq 1$, $f \in B^0_{M_\beta, w_\alpha}$ if and only if*

$$\max\Big(|f_0|, |f_1|, \sup_{p \geq p_0} \sup_{j \geq 0} 2^{-j(\frac{1}{2}-\alpha+\frac{1}{p})} p^{-\gamma}(1+j)^{-\lambda} \parallel f_{j,.} \parallel_p \Big) < \infty \tag{4}$$

Theorem 3.4. *Let $f \in B^0_{M_\beta, w_\alpha}$ if and only if*

$$\lim_{j \vee p \rightarrow p_0} 2^{-j(\frac{1}{2}-\alpha+\frac{1}{p})} p^{-\gamma}(1+j)^{-\lambda} \parallel f_{j,.} \parallel_p \Big) < \infty \tag{5}$$

Consider the following norms that are crucial to proving our results :

$$\parallel f \parallel_{**} = \sup_{0 \leq s < t \leq 1} \frac{|f(t) - f(s)|}{w(t-s)}$$

this is dominated by

$$\parallel f \parallel_{*} = \max\Big(|f(1)|, \sup_{j \geq 0} \sup_{0 \leq k \ \leq 2^j} \frac{|f_{j,k}|}{\sqrt{1+j}}\Big).$$

It's easy to show that there exist $D_1 > 0$ and $D_2 > 0$ such that $\parallel f \parallel_{M_2,w} \leq D_1 \parallel f \parallel_{**} \leq D_2 \parallel f \parallel_*$.

Theorem 3.5. *Le P^ε be the law of $\sqrt{\varepsilon}W$ on $B^0_{M_2,w_\alpha}$ equipped with the norm $\|\|_{M_\beta,w_\alpha}$, then P^ε satisfy the LDP with the good rate function I define by :*

$$I(f) = \begin{cases} \frac{1}{2}\int_0^T |\dot{f}(s)|^2\, ds & \text{if } f \in H(\mathbb{R}^d) \\ +\infty & \text{other} \end{cases}$$

Theorem 3.6. *Let Q^ε be a family of probability measure on a Polish space E and satisfying the LDP with a good rate function λ.*
Let $F : E \longrightarrow E'$ be countinuous. Denote by $Q^\varepsilon = P^\varepsilon \circ F^{-1}$ the family of image measure of P^ε, then $\{Q^\varepsilon\}$ satisfy the LDP with a good rate function $\tilde{\lambda}$ define by

$$\tilde{\lambda}(y) = \inf_{x:f(x)=y} \lambda(x).$$

Lemma 3.7. *There exist $C = C_l$ such that for all $\lambda > 0$ and $\mu > 0$ where $\lambda > 4l\mu > 0$ and $\lambda > 2\sqrt{\log 2}$, we have*

$$P\Big[\parallel W \parallel_{**} \geq \lambda, \parallel W \parallel \leq \mu \Big] \leq C \max\left(1, l\Big(\frac{\lambda}{4l\mu}\Big)^2 \exp\Big(-\frac{\lambda^2}{C}\ln\Big(\frac{\lambda}{4l\mu}\Big)\Big)\right) \tag{6}$$

Lemma 3.8. *There exist $C = C_l$ such that for all $u > 2\sqrt{\log 2}$ and for all process K on $[0, 1]$, we have*

$$P\Big[\parallel \int_0^{\cdot} K_s\, dW_s \parallel_{**} \geq u, \parallel K \parallel \leq 1 \Big] \leq C \exp\Big(-\frac{u^2}{C}\Big). \tag{7}$$

Lemma 3.9. *Let $(E_X, d_X), (E_Y, d_Y), (E_Z, d_Z), (E, d)$ denote Polish spaces and (Ω, F, P) be a probability space.*

Suppose that $(X^\varepsilon, \varepsilon > 0)$ is a family of random variables with values in E_X satisfy a LDP with a good rate function I_X, $(Y^\varepsilon, \varepsilon > 0)$ a random variable with values in E_Y satisfy a LDP with a good rate function I_Y.

Suppose that for each $\varepsilon > 0$, X^ε is independant of Y^ε then the family of random variable $Z = F(X^\varepsilon, Y^\varepsilon)$ where $F : E_X \times E_Y \longrightarrow E_Z$ is continues, satisfy a LDP with good rate function $I_F(z)$ define by

$$I_F(z) = \inf_{F(x,y)=z} I_X(x) + I_Y(y).$$

4 The main result

Let $\varphi \in \mathcal{C}([0,1],\mathbb{R}^d)$, and $X^{\varepsilon,\varphi}$ solution of SDE

$$dX_t^{\varepsilon,\varphi} = b(X_t^{\varepsilon,\varphi}, \xi_{t/\varepsilon})\ dt + \sqrt{\varepsilon}\sigma(X_t^{\varepsilon,\varphi})\ dW_t. \tag{8}$$

Let φ absolutely continuous, the function F^{φ} with values in $\left(\mathcal{C}([0,1],\mathbb{R}^d)\right)^2$ by

$$F^{\varphi}(g,f) = h \ \text{ if and only if } h_t = x + g_t + \sigma(\varphi_t)f_t + \int_0^t f_s d\sigma(\varphi_s)$$

is continue.
$X^{\varepsilon,\varphi}$ is the family process image of $(y^{\varepsilon,\varphi}, \sqrt{\varepsilon}W)$ by F^{φ}.
and $dy_t^{\varepsilon,\varphi} = b(y_t^{\varepsilon,\varphi}, \xi_{t/\varepsilon})\ dt \quad , \quad y_0^{\varepsilon,\varphi} = 0.$
Let g, f be given an elements of $B^0_{M_2,w_\alpha}$ with values in $\mathbb{R}^d$ absolutly continuous, denote by $B_x(g,f)$ the solution of $\dot{\varphi}_t = \dot{g}_t + \sigma(\varphi)\dot{f}_t, \varphi_0 = x$.
Let $L^0(x,\alpha)$ the conjugate of the quadratic convex function $H^0(x,\alpha)$ obtained from the formula in (3). L^0 is lower semicontinuous(lsc), with values in $\mathbb{R}_+ \cup \{\infty\}$, convex to second argument

For some couple values (φ,ψ) in $B([0,T],\mathbb{R}^d)$, denote by :

$$\begin{cases} S^0(\varphi,\psi) & = \int_0^T L^0(\varphi_s,\dot{\psi}_s)\ ds \ \ \textit{if } \psi \textit{ is absolutly continuous} \\ & = +\infty \ \ \textit{other} \end{cases}$$

$$\begin{cases} S^W(\psi) & = \int_0^T \frac{1}{2} \mid \dot{\psi}_s \mid^2 \ ds \ \ \textit{if } \psi \textit{ is absolutly continuous} \\ & = +\infty \ \ \textit{other} \end{cases}$$

Proposition 4.1. *Let $\varphi \in B_{M_2,w_\alpha}$. let $y^{\varepsilon,\varphi}(0)$ be the solution of $dy_t^{\varepsilon,\varphi} = b(y_t^{\varepsilon,\varphi}, \xi_{t/\varepsilon})\ dt$ starting from 0, then $S^0(\varphi,.)$ is a rate function for the law of $y^{\varepsilon,\varphi}(0)$ in $B^0_{M_2,W_\alpha}$.*

Proposition 4.2. *The independance of ξ and W, and using lemma 3.9, it is easy to see that $(y^{\varepsilon,\varphi}(0), \sqrt{\varepsilon}W)$ satisfy LDP and $S^{\varphi}(g,f)$ is a rate function in $B^0_{M_2,w_\alpha}$ definie by :*

$$S^{\varphi}(g,f) = S^0(\varphi,g) + S^W(f)$$

.

By the contraction principle, the law of X^{ε} satisfy LDP on $B^0_{M_2,w_\alpha}$ with the rate function defined by :

$$S\varphi(\omega) = \inf\{S^{\varphi}(g,f), \omega = F^{\varphi}(g,f)\}.$$

Theorem 4.3. *Assume* **H1**, **H2**, **H3** *et* **H4** *are satisfed, let* P_ε *the law of* X^ε *solution of* (1), X^ε *is a random variable in* $B^0_{M_2,w_\alpha}$.

i) For each positive α, $K_\alpha = \{\varphi \in B^0_{M_2,w_\alpha} / S_\varphi(\psi) \leq \alpha\}$
ii) For every open subset O *of* $B^0_{M_2,w_\alpha}$,

$$\overline{\lim_{\varepsilon \to 0}}\, \varepsilon \log P(X^\varepsilon(x) \in O) \geq - \inf_{\varphi \in O} S^{(\psi)}_\varphi.$$

iii) For every closed F *of* $B^0_{M_2,w_\alpha}$,

$$\overline{\lim_{\varepsilon \to 0}}\, \varepsilon \log P(X^\varepsilon(x) \in F) \leq - \inf_{\varphi \in F} S^{(\psi)}_\varphi.$$

where

$$S_\varphi(\psi) = \inf\{s^\varphi(g,f), \psi = F^\varphi(g,f)\}.$$

Theorem 4.4. *For any* $r, \alpha, a > 0$, *for each* x *with values in* $\mathbb{R}^d$, *there exist* $\rho, \tilde{r}, \varepsilon_0$ *depending only on* r, α, a, x *such that for* g, f *absolutly continuous verifing* $\| \dot{f} \| \leq a$ *and* $\varphi = B_y(g,f)$, $|x-y| \leq \tilde{r}$, $\varepsilon \leq \varepsilon_0$ *we have,*

$$\mathbb{P}\Big(\| X^\varepsilon(x) - \varphi \|_{M_2,w_\alpha} > \alpha, \| y^{\varepsilon,\Phi}(0) - g \| < \rho, \| \sqrt{\varepsilon} W - f \| < \rho \Big) \leq \exp(-\frac{r}{\varepsilon}).$$

Proof of theorem 4.4. Indeed, let $W^f = W - \frac{1}{\sqrt{\varepsilon}} f$ be a brownian motion starting from 0. Girsanov's theorem implies that W^f be the standard k-dimensional Wiener process with respect to the probability P^f given by

$$\frac{dP^f}{dP} = \exp\Big(\frac{1}{\sqrt{\varepsilon}} \int_0^1 \dot{f}_s\, dW_s - \frac{1}{\varepsilon} \int_0^1 |\dot{f}_s|^2\, ds\Big)$$

Let $\{Y^\varepsilon_t, 0 \leq t \leq 1\}$ the solution of SDE

$$Y^\varepsilon_t = \int_0^t b(Y^\varepsilon_s, \zeta_{s/\varepsilon})\, ds + \sqrt{\varepsilon} \int_0^t \sigma(Y^\varepsilon_s)\, dW^f_t + \int_0^t \sigma(X^\varepsilon_s) \dot{f}_s\, ds \tag{9}$$

To simplify the notation, set for any ρ ,α, $\varepsilon > 0$

$$U^f = \{\| X^\varepsilon(x) - \varphi \|_{M_2,w_\alpha} > \alpha, \| y^{\varepsilon,\Phi}(0) - g \| < \rho, \| \sqrt{\varepsilon} W - f \| < \rho\}$$

And

$$V^f = \exp\Big\{\Big| \frac{1}{\sqrt{\varepsilon}} \int_0^1 \dot{f}_s\, dW_s \Big| > \frac{\lambda}{\sqrt{\varepsilon}} \Big\}.$$

Then

$$\begin{aligned} P(U^f) &\leq P\Big\{U^f \cap \Big(V^f \leq \exp\left(\tfrac{\lambda}{\varepsilon}\right)\Big)\Big\} + P\Big\{V^f > \tfrac{\lambda}{\varepsilon}\Big\} \\ &\leq \exp\left(\tfrac{\lambda + a/2}{\varepsilon}\right) P^f(U^f) + P\Big(\Big|\tfrac{1}{\sqrt{\varepsilon}} \textstyle\int_0^1 \dot{f}_s \, dW_s\Big| \geq \tfrac{\lambda}{\varepsilon}\Big) \end{aligned} \tag{10}$$

Where $a = \| h \|_H^2$ and $\lambda \in \mathbb{R}$
By the classical exponential inequality,

$$P\Big(\Big|\int_0^1 \dot{f}_s \, dW_s\Big| \geq \frac{\lambda}{\sqrt{\varepsilon}}\Big) \leq 2\exp(-\frac{\lambda^2}{2a\varepsilon}) \leq \exp(-\frac{r}{\varepsilon}). \tag{11}$$

Set

$$Y^\varepsilon(W^f) = X^\varepsilon(W^f + \frac{1}{\sqrt{\varepsilon}} f).$$

Consequently, we obtain that

$$P^f(U^f) = P\Bigg(\| Y^\varepsilon(x) - \varphi \|_{M_2, w_\alpha} > \alpha, \| y^{\varepsilon,\Phi}(0) - g \| < \rho, \| \sqrt{\varepsilon} W \| < \rho \Bigg),$$

where Y^ε is the solution of SDE in (9), the estimates (10) and (11) complete the proof of the theorem (4.4).

The remains of proof of theorem 4.4 is an immediate consequence of the next following propositions.

Proposition 4.5. *For all $r > 0$ and $\gamma > 0$ there exist $\varepsilon_0 > 0$ and n such that if $0 < \varepsilon < \varepsilon_0$, we have :*

$$P^f(\Big\{ \| X^\varepsilon - X^{\varepsilon,n} \| \geq \gamma \Big\}) \leq \exp(-\frac{r}{\varepsilon})$$

Proof of Proposition 4.5. For a detailed proof of Proposition 4.5, we refer to Priouret,P(1982, Lemma 2) [21]

Proposition 4.6. *For every $\gamma_1 > 0$, $\rho > 0$ on a :*

$$P^f(U^f) \leq P^f(\| \sqrt{\varepsilon} \int_0^{\cdot} \sigma(Y_s^\varepsilon) \, dW_s^\varepsilon \|_{**} > \gamma_1, \| \sqrt{\varepsilon} W^\varepsilon \| < \rho).$$

Proof of proposition 4.6.

$$\begin{aligned} Y_t^\varepsilon - \varphi &= x - y + \int_0^t [b(Y_s^\varepsilon, \xi_{s/\varepsilon}) + \sigma(Y_s^\varepsilon)\dot{f}_s] \, ds + \sqrt{\varepsilon} \int_0^t \sigma(Y_s^\varepsilon) \, dW_s^\varepsilon \\ &\quad - \int_0^t [b(\varphi_s, \xi_{s/\varepsilon}) + \sigma(\varphi_s)\dot{f}_s] \, ds + y_t^{\varepsilon,\varphi} - g_t \\ &= x - y + \int_0^t [b(Y_s^\varepsilon, \xi_{s/\varepsilon}) - b(\varphi_s, \xi_{s/\varepsilon})] \, ds + \int_0^t [\sigma(Y_s^\varepsilon) + \sigma(\varphi_s)]\dot{f}_s \, ds \\ &\quad + \sqrt{\varepsilon} \int_0^t \sigma(Y_s^\varepsilon) \, dW_s^\varepsilon + y_t^{\varepsilon,\varphi} - g_t \end{aligned}$$

Denote by $I_t^\varepsilon = \sqrt{\varepsilon}\int_0^t \sigma(Y_s^\varepsilon)\ dW_s^\varepsilon$, let $\delta > 0$ be such that $\|I_t^\varepsilon\| \leqslant \delta, \|x-y\| \leqslant \tilde{r}$

$$\begin{aligned} \|Y_t^\varepsilon - \varphi\| &\leqslant \tilde{r} + C\int_0^t |Y_s^\varepsilon - \varphi_s|\ ds + C\int_0^t |Y_s^\varepsilon - \varphi_s|\ |\dot{f}_s|\ ds + \|I_t^\varepsilon\| + \|y_t^{\varepsilon,\varphi} - g_t\| \\ &\leqslant \tilde{r} + C\int_0^t |Y_s^\varepsilon - \varphi_s|(1+|\dot{f}_s|)\ ds + \|I_t^\varepsilon\| + \|y_t^{\varepsilon,\varphi} - g_t\| \end{aligned}$$

An application of Gronwall's lemma implies that,

$$|Y_t^\varepsilon - \varphi_t| \leq (\tilde{r}+ \parallel y^{\varepsilon,\varphi} - g \parallel + \parallel \sqrt{\varepsilon}\int_0^\cdot \sigma(Y_s^\varepsilon)\ dW_s^\varepsilon \parallel) \exp\Big(C(\int_0^t (1+|\dot{f}_s|\ ds))\Big).$$

On one hand

$$\begin{aligned} \parallel X_t^\varepsilon - \varphi_t \parallel_{**} &\leq \parallel \sqrt{\varepsilon}\int_0^\cdot \sigma(X_s^\varepsilon)\ dW_s^\varepsilon \parallel_{**} + \parallel y^{\varepsilon,\varphi} - g \parallel \\ &\quad + \parallel \int_0^t [b(X_s^\varepsilon, \xi_{s/\varepsilon}) + \sigma(X_s^\varepsilon) f_s]\ ds - \int_0^t [b(\varphi_s, \xi_{s/\varepsilon}) + \sigma(\varphi_s) f_s]\ ds \parallel_{**} \\ &\leq \|\sqrt{\varepsilon}\int_0^\cdot \sigma(X_s^\varepsilon)\ dW_s^\varepsilon\|_{**} + \|y^{\varepsilon,\varphi} - g\| + \|\int_0^t [b(X_s^\varepsilon, \xi_{s/\varepsilon}) - b(\varphi_s, \zeta_{s/\varepsilon})]\ ds\|_{**} \\ &\quad + \|\int_0^t [\sigma(X_s^\varepsilon) - \sigma(\varphi_s)]\dot{f}_s\ ds\|_{**} \\ &\leq \|I_t^\varepsilon\|_{**} + \|y^{\varepsilon,\varphi} - g\| + C\int_0^t \|X_s^\varepsilon - \varphi_s\|_{**}(1+|\dot{f}_s|)\ ds \\ &\leq \|I_t^\varepsilon\|_{**} + \|y^{\varepsilon,\varphi} - g\| + \sup_{0\leq u\leq v\leq 1} \frac{C}{w(u-v)}\int_u^v (1+|\dot{f}_s|)|X_s^\varepsilon - \varphi_s|\ ds \end{aligned}$$

On the other hand, using the fact that

$$|Y_s^\varepsilon - \varphi_s| \leq |Y_u^\varepsilon - \varphi_u| + |(Y_s^\varepsilon - \varphi_s) - (Y_u^\varepsilon - \varphi_u)|$$

$$\begin{aligned} \|Y^\varepsilon - \varphi\| &\leq \|I_s^\varepsilon\| + \|y^{\varepsilon,\varphi} - \varphi\| + C(1+|f|)\|X^\varepsilon - \varphi\| + C\int_0^t (1+|\dot{f}_s|)\|X^\varepsilon - \varphi\|_{**}\ ds \\ &\leq 2\delta + C(1+|f|)\|X^\varepsilon - \varphi\| + C\int_0^t (1+|\dot{f}_s|)\ ds \end{aligned}$$

Using now Grownall's lemma, we obtain

$$\|Y^\varepsilon - \varphi\| \leq 2\delta\Big[1 + C(1+|f|)e^{C(1+|f|)}\Big]e^{C(1+|f|)}$$

Thus :

$$P^f(U^f) \leq P^f(\parallel \sqrt{\varepsilon}\int_0^\cdot \sigma(Y_s^\varepsilon)\ dW_s^\varepsilon \parallel_{**} > \gamma_1, \parallel \sqrt{\varepsilon}W^\varepsilon \parallel < \rho).$$

Proposition 4.7. *For all $r > 0, \gamma_1 > 0$, there exist $\varepsilon > 0$ and $\rho > 0$ such that*

$$P^f(\| \sqrt{\varepsilon} \int_0^. \sigma(Y_s^\varepsilon)\ dW_s^\varepsilon \|_{**} > \gamma_1, \| \sqrt{\varepsilon} W^\varepsilon \| < \rho) \leq \exp(-\frac{r}{\varepsilon}).$$

Proof of proposition 4.7. For $\alpha > 0$ and for every $n \in \mathbb{N}$, we have

$$A = \Big\{ \| \sqrt{\varepsilon} \int_0^. \sigma(Y_s^\varepsilon)\ dW_s^\varepsilon \|_{**} \geq \rho, \| \sqrt{\varepsilon} W \| \leq \alpha \Big\} \subset A_1 \cup A_2 \cup A_3$$

where

$$\begin{cases} A_1 = \Big\{ \| \sqrt{\varepsilon} \int_0^. [\sigma(Y_s^\varepsilon) - \sigma(Y_s^{\varepsilon,n})] dW_s^\varepsilon \|_{**} \geq \frac{\rho}{2}, \| Y^\varepsilon - Y^{\varepsilon,n} \| \leq \gamma \Big\} \\ A_2 = \Big\{ \| Y^\varepsilon - Y^{\varepsilon,n} \| \geq \gamma \Big\} \\ A_3 = \Big\{ \| \sqrt{\varepsilon} \int_0^. \sigma(Y_s^{\varepsilon,n})\ dW_s \|_{M_2,w} \geq \frac{\rho}{2}, \| \sqrt{\varepsilon} W \| \leq \alpha \Big\} \end{cases}$$

By using Proposition 4.5, we have :
For all $r > 0$ and $\gamma > 0$ there exist ε_0 and n such that for every $0 < \varepsilon < \varepsilon_0$, we have :

$$P^f(A_2) \leq \exp(-\frac{r}{\varepsilon})$$

It's easy to check that if $\| Y^\varepsilon - Y^{\varepsilon,n} \| \leq \gamma$ we get $\| \sqrt{\varepsilon}[\sigma(Y_s^\varepsilon) - \sigma(Y_s^{\varepsilon,n})] \|_{**} \leq 4\varepsilon M^2 \gamma^2$.
By using the lemma (3.8),

$$P^f(A_1) \leq C \exp\Big(- \frac{\rho^2}{C\gamma^2\varepsilon}\Big)$$

It remains to be increased $P^f(A_3)$. So we have

$$\begin{aligned} \| \sqrt{\varepsilon} \int_0^. \sigma(Y_s^{\varepsilon,n})\ dW_s \|_{M_2,w} &= \sqrt{\varepsilon} \| \sum_{j=0}^n \sigma(Y_{t_j}^{\varepsilon,n})[W(t_{j+1} \wedge .) - W(t_j \wedge .)] \|_{M_2,w} \\ &\leq \sqrt{\varepsilon} \sum_{j=0}^n \| \sigma(Y_{t_j}^{\varepsilon,n})[W(t_{j+1} \wedge .) - W(t_j \wedge .)] \|_{M_2,w} \\ &\leq 2\sqrt{\varepsilon} K n \| W \|_{**}\ . \end{aligned}$$

By using the lemma (3.7), we have :

$$P^f(A_3) \leq C \max\Bigg(1, \Big(\frac{\rho}{16lMn\alpha}\Big)^2\Bigg) \exp\Big(- \frac{\rho^2}{C\varepsilon 16M^2n^2} \log\Big(\frac{\rho}{16lMn\alpha}\Big)\Big)$$

where C is a constant depending on l et M.
Let $r > 0$ et $\rho > 0$, we choose then $\gamma > 0$ small enough that $\frac{\rho}{C\gamma^2} > r$, and n such that

$$P^f(A_1) \leq C \exp\Big(- \frac{r}{\varepsilon}\Big)$$

and finally $\Big(\frac{\rho^2}{16M^2n^2} \log\Big(\frac{\rho}{16lMn\alpha}\Big)\Big) > Cr$ in (12). This ends the proof of the proposition.

4.1 Construction of the rate function

For any $(x,\alpha) \in (\mathbb{R}^d)^2$, denote by $H(x,\alpha) = H^0(x,\alpha) + \frac{1}{2}\langle \alpha, \Sigma_x \alpha\rangle$ the quadratic function associated of $\sigma(x)$ where $\Sigma_x = \sigma(x)^t\sigma(x)$.
Let us suppose that $L(x,\beta)$ the conjugate quadratic function $H(x,\alpha)$ en α. L is lower semicontinuous with values $\mathbb{R}_+ \cup \{+\infty\}$, converges en β, and verified the following :
For all $\varphi, \psi \in B([0,T], \mathbb{R}^d)$:

$$S(\varphi,\psi) = \begin{cases} \int_0^T L(\varphi_s, \psi_s)\, ds, & \text{if } \psi \text{ is absolutly continuous,} \\ +\infty & \text{sinon.} \end{cases}$$

Theorem 4.8. *$S(\varphi,.)$ and S^φ coincident and if $S(\varphi,\psi) < +\infty$, there exist a couple of functions absolutly continuous (g,f) verifing $\psi = F^\varphi(g,f)$ and we have : $S(\varphi,\psi) = S^0(\varphi,\psi) + S^W(f)$.*

Proof of theorem 4.8. We denote for $(x,\alpha) \in (\mathbb{R}^d)^2$, $H(x,\alpha) = H^0(x,\alpha) + \frac{1}{2}\langle \alpha, \Sigma_x \alpha\rangle$. Q_x denotes the quadratic form on $\mathbb{R}^n$ associated with the matrix $\sigma(x)$, defined by $Q_x(v) = \langle v, \sigma(x)\sigma(x)^* v\rangle = \inf |w|^2,\ \sigma(x)w = v,\ v \in \mathbb{R}^n$.
We denote for $(x,\beta) \in (\mathbb{R}^d)^2$,

$$L(x,\beta) = \inf\{L^0(x,\gamma) + Q^*(\delta); b(\gamma) + \delta = \beta\}$$

where Q^* is the quadratic form Q_x.
Let $\tau = B_x(g,f)$ the solution of $\dot{\tau}_t = b(\dot{g}_t) + \sigma(\tau_t)\dot{f}_t$

$$\begin{aligned} S^0(\varphi,g) + S^W(f) &= \int_0^T L^0(\varphi_s, \dot{g}_s) + \frac{1}{2}|\dot{f}_s|^2\, ds \\ &\geqslant \int_0^T L^0(\varphi_s, \dot{g}_s) + \frac{1}{2}\inf\{|\dot{g}_s|^2; \sigma(\tau_s)\dot{g}_s \nabla_s\}\, ds \\ &\geqslant \int_0^T L^0(\varphi_s, \dot{g}_s) + \frac{1}{2}Q^*_{\varphi_s}(\nabla_s)\, ds \\ &\geqslant \int_0^T \inf\{L^0(\varphi_s, \dot{g}_s) + \frac{1}{2}Q^*_{\varphi_s}(\nabla_s); b(\dot{g}_s) + \nabla_s = \dot{\tau}_s\}\, ds \\ &\geqslant \int_0^T L^0(\varphi_s, \dot{\tau}_s)\, ds \end{aligned}$$

It follows that,

$$S^\varphi(\tau_s) \geqslant S(\varphi,\tau).$$

To check the other inequality, consider $A_x[v]$ defined by

$$A_x[v] = \{w \text{ tel que } \sigma(x)w = v,\ v \in \mathbb{R}^n\}$$

Consider the Borel set Γ defined by

$$\Gamma = \{(x, v) \in U \times \mathbb{R}^n \text{ such that } A_x[v] \text{ is not empty}\}$$

For each $(x, v) \in \Gamma$, we put

$$K(x, v) = \{w \in \mathbb{R}^n \text{ tel que } |w| = \inf |u|; u \in A_x[v]\}$$

The application $K : \Gamma \to \{\text{compacts of } \mathbb{R}^k\}$ is a measurable family of non-empty compacts in the sense of Rockafeller [25]. Subsequently, there is a Borelian function $\chi : \Gamma \to \mathbb{R}^k$ such that $\chi(x, v) \in K(x, v)$ for $(x, v) \in \Gamma$.
For each φ, ψ such that $S(\varphi, \psi) < +\infty$ and we put Ω the set of (x, β) such that $L(x, \beta) < +\infty$

$$S(\varphi, \psi) = \int_0^T L(\varphi_s, \psi_s)\, ds$$

As

$$Q_x(v) = \langle v, \sigma(x)\sigma(x)^* v\rangle = \|\sigma^*(x)v\|^2$$

and

$$Q_x^*(v) = \inf\{|w|^2, w \in A_x[v]\},$$

we have

$$Q^*_{\varphi_s}(\varphi'_s - b(\varphi_s)) = |\chi(\varphi_s, \varphi'_s - b(\varphi_s))|^2$$

$$S(\varphi, \psi) = \int_0^T L(\varphi_s, \psi_s)\, ds = \int_0^T \inf\{L^0(\varphi_s, \dot{g}_s) + \frac{1}{2} Q^*_{\varphi_s}(\nabla_s); b(\dot{g}_s) + \nabla_s = \dot{\tau}_s\}\, ds.$$

So there exist a functional $f \in C^0(\mathbb{R}^k)$ such that

$$S(\varphi, \psi) \leqslant \int_0^T \inf\{L^0(\varphi_s, \dot{g}_s) + \frac{1}{2}|\dot{f}|^2\}\, ds.$$

It suffices to ask $\dot{f}_s = |\chi(\varphi_s, \nabla_s)|$ for almost everything $s \in [0, T]$.

4.2 Regularity of the solution in the Besov-Orlicz space

It is clear that the process $\int_0^t b(X_s^\varepsilon, \zeta_{s/\varepsilon})\, ds, t \in I$ belongs a.s. to $B^{\varphi,0}_{M_2,w}$. Then, it remains to show that the process $\int_0^t \sigma(X_s^\varepsilon)\, dW_s, t \in I$ satisfies (1.1) and (1.2). We will prove the result in the case $k = d = 1$.
Let us put

$$Y_t = \int_0^t \sigma(X_s^\varepsilon)\, dW_s$$

We will show that for some p_0, we have for any $\alpha < \frac{1}{2}$

$$\sup_{j\geqslant 0}\sup_{p\geqslant p_0}\frac{2^{-j/p}}{p^{1/2}(1+j)^{\alpha}}\Big[\sum_{n=2^j+1}^{2^{j+1}}|A_n(Y)|^p\Big]^{1/p}<\infty \quad p.s. \tag{12}$$

$$\lim_{j\vee p\to p_0}\frac{2^{-j/p}}{p^{1/2}(1+j)^{1/2}}\Big[\sum_{n=2^j+1}^{2^{j+1}}|A_n(Y)|^p\Big]^{1/p}=0 \tag{13}$$

To check relation 12, let $\lambda > 0$. Using Chebychev inequality, we get

$$P\Big(\frac{2^{-j/p}}{p^{1/2}(1+j)^{\alpha}}\Big[\sum_{n=2^j+1}^{2^{j+1}}|A_n(Y)|^p\Big]^{1/p}>\alpha\Big)\leqslant\frac{\lambda^{-p}\,2^{-j}}{\sqrt{\frac{p}{2}}(1+j)^{\alpha p}}\Big(\sum_{n=2^j+1}^{2^{j+1}}|A_n(Y)|^p\Big)$$

$|A_n(Y)|$ is dominated by terms of the form

$$A:=\Big|\int_0^t f_{\frac{2k-1}{2^{j+1}},\frac{2k}{2^{j+1}}}(s)\,dW_s\Big| \quad \text{et} \quad B:=\Big|\int_0^t f_{\frac{2k-2}{2^{j+1}},\frac{2k-1}{2^{j+1}}}(s)\,dW_s\Big|,$$

where

$$f_{r,t}(s)=1_{r<s\leqslant t}\sigma(t,X_s)+1_{s<r\leqslant t}[\sigma(t,X_s)-\sigma(r,X_s)].$$

For integers $p \geqslant 2$, using the inequality of Barlow-Yor(1982), for A and B, there exist a constant C_p appearing in the Burkholder-Davis-Gundy inequality such that

$$E|A_n(Y)|^p\leqslant CM^pp^{p/2}.$$

Hence,

$$\begin{aligned}P\Big(\frac{2^{-j/p}}{p^{1/2}(1+j)^{\alpha}}\Big[\sum_{n=2^j+1}^{2^{j+1}}|A_n(Y)|^p\Big]^{1/p}>\alpha\Big) &\leqslant \frac{\lambda^{-p}\,2^{-j}}{\sqrt{\frac{p}{2}}(1+j)^{\alpha p}}\Big(\sum_{n=2^j+1}^{2^{j+1}}|A_n(Y)|^p\Big)\\ &\leqslant \Big(\frac{C}{\lambda}\Big)^p\frac{1}{(1+j)^{\alpha p}}\end{aligned}$$

Choosing $p_0 \geqslant \frac{1}{\alpha}$ and λ large enough, the series

$$\sum_{j\geqslant 0}\sum_{p\geqslant p_0}\Big(\frac{C}{\lambda}\Big)^p\frac{1}{(1+j)^{\alpha p}}$$

converges. The point (12) is then a consequence of Borel-Cantelli lemma.
To prove 13, let us remark that as above $|A_n(Y)|$ is dominated by terms of the

form A et B the exponential inequalities yields that there exist positive constants K_1 et K_2 such that for all $\lambda > 0$ large enough,

$$P\Big(\frac{1}{\sqrt{1+j}}\sup_n |A_n(Y)| > \alpha\Big) \leqslant K_1 \exp\frac{-\lambda^2(1+j)}{K_2 M^2}\ .$$

Therefore, the Borel-Cantelli lemma leads to

$$\sup_{j\geqslant 1}\frac{1}{\sqrt{1+j}}\sup_n |A_n(Y)| < \infty \quad p.s.$$

Or

$$2^{-j/p}\Big[\sum_{n=2^j+1}^{2^{j+1}} |A_n(Y)|^p\Big]^{1/p} \leqslant \sup_n |A_n(Y)|$$

Thus

$$\sup_{j\geqslant 1}\frac{2^{-j/p}}{p^{1/2}(1+j)^{1/2}}\Big[\sum_{n=2^j+1}^{2^{j+1}} |A_n(Y)|^p\Big]^{1/p} \leqslant \frac{1}{p^{1/2}}\sup_{j\geqslant 1}\sup_n |A_n(Y)|.$$

and this ends the establishment of (13).

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