Original Research Article

A New Compound Family of Generalized Moment Exponential distribution and Power Series Distribution~~: Properties and Applications~~

ABSTRACT

This paper introduces a family of distributions based on generalized moment exponential power series (GMEPS) distribution which is a general form of the moment exponential power series (MEPS) distribution proposed by Sadaf (2014). This new family is developed through compounding generalized moment exponential (GME) distribution and truncated power series (PS) distributions. This new family have some new sub models such as GME geometric distribution, GME Poisson (GMEP) distribution, GME logarithmic (GMEL) distribution and GME binomial (GMEB) distribution. Properties of GMEPS family of distributions are studied, among them; quantile function, order statistics, moments and entropy. Some special models in the GMEPS family of distributions are provided. The estimates of parameters of GMEPS distribution are obtained through maximum likelihood (ML) method is applied to obtain and a simulation study is conducted to check the convergence of ML estimators of the parameters of GMEG distributions. To check validity of these distributions, two sets of real data are used and the results demonstrate that the sub-models from the GMEPS family can be considered as suitable models under several real situations.

KEYWORDS

Hazard rate function, generalized moment exponential distribution; power series distribution; order statistics.

1. INTRODUCTION

The problem of finding a suitable model for the real life data has been studied extensively in literature, however, there are many situations where existing models are not suitable or less representative of real data.~~, therefore, as a result~~ To resolve this situation one needs to develop a general model. The well-known and existed distributions are very limited in their characteristics, for example the distributions: exponential, Rayleigh, Weibull, gamma and beta are unable to show wide flexibility in modeling many real situations. In 1997, some authors started the use of shape parameter(s) for the purpose of generalization of any probability distribution and such techniques are continuously in practice from the last two decades. In literature, various distributions through compounding lifetime distributions with discrete distribution have been discussed to model lifetime data. Compounding lifetime distributions have been obtained by mixing up the distribution when the lifetime can be expressed as the minimum (maximum) of a sequence with a discrete random variable. This idea was first pioneered by Adamidis and Loukas (1998) and they compounding the exponential random variable simultaneously with a geometric random variable. Several authors introduced new lifetime distributions (see for example; are introduced,(Kus (2007), Barreto-Souza et al. (2011), and Lu and Shi (2012)).

In recent years, a great effort has been made to define new compounding families of distributions by mixing lifetime distributions with power series distributions. The new families extend some compound distributions and yield more flexibility in modeling several practical data. Some authors defined new families of lifetime distributions (see for example; exponential-power series (PS) distribution [See Chahkandi and Ganjali; 2009] , Weibull-PS distributions [See Morais and Barreto-Souza; 2011], generalized exponential PS distribution [Mahmoudi and Jafari ; 2012], extended Weibull PS distribution [See Silva et al. ; 2013] Burr XII PS distribution [See Silva and Corderio ; 2015],

The moment exponential (ME) (or length biased) distribution was proposed by Dara (2012) and discussed hazard and reversed hazard rate functions. The ME distribution has the *pdf* as:

$$g(y;\beta)=\beta^2 y e^{-\beta y}, \quad y,\beta>0. \tag{1}$$

It is also called gamma distribution $G(2,\beta)$. Followed the technique generalizing a distribution used by iqbal et al. (2013), the *pdf* of the generalized moment exponential distribution is derived by Sadaf (2014), after applying transformation $Y=X^{\alpha}$, in (1) as

$$g(x;\alpha,\beta)=\alpha\beta^2 x^{2\alpha-1} e^{-\beta x^{\alpha}}, \quad x,\alpha,\beta>0. \tag{2}$$

Also, a discrete r.v. *Z* is a family member of PS distributions which is truncated at zero and pmf of Z is:

$$P(Z=z;\theta)=\frac{a_z\theta^z}{K(\theta)}, \; z=1,2,3..., \tag{3}$$

where, $\theta>0$ is the scale parameter. The coefficients a_z's depend only on z, $K(\theta)=\sum_{z=1}^{\infty} a_z\theta^z$ is finite, $K'(.)$ and $K''(.)$ denote its first and second derivatives, respectively. Noack (1950) derived (3) and this family contains some well-known PS family of distributions such as the binomial, geometric, logarithmic, negative binomial and Poisson distributions.

In this article, a quite flexible family of distributions based on *GMEPS* distributions is introduced and applied on positive data and we find here some of its properties which will show wider applications in the research areas of reliability and engineering. The *GMEPS* family of distributions permit flexibility in a real data modeling. We shall see that the *GMEPS* family distributions allow for different hazard shapes i.e. increasing or decreasing or bathtub (increasing or decreasing) failure rates. We shall also see later that the *GMEG* i.e. member of *GMEPS* family distributions provides significantly better fits than Weibull, exponential and exponentiated exponential distributions for two data sets.

The contents of the remaining part of this paper is arranged as follows: Section 2 deals with derivation of *GMEPS* distribution, cumulative, survival and hazard rate functions of *GMEPS* family distributions. In the following section 3, some Statistical properties like quantile, moments, entropy and order statistics are presented. Section 4 related to some special sub-models of *GMEPS* distribution. In Section 5, maximum likelihood (ML) estimators for the unknown parameters on the basis of the family are obtained and a simulation study is carried out on the basis of ML estimates and of method of moments. In Section 6, *GMEG* distribution is applied on two data sets [Murthy et al.;2004, Bjerkedal ;1960] and comparison is made with reputed lifetime models via statistical analysis which show the flexibility and applicability of the proposed family of distributions**.** Finally, Section 7 is devoted for some concluding remarks.

2. NEW FAMILY OF DISTRIBUTIONS

In this section, the GMEPS family of distributions is proposed. This new family is derived after compounding the generalized ME distribution and PS distributions.

Let $X_1, X_2, ..., X_z$ be iid r.v's having *GME* distribution with pdf (1) and the following cdf:

$$G(x;\alpha,\beta) = 1 - H(x;\alpha,\beta) \text{ where } H(x;\alpha,\beta) = (1+\beta x^{\alpha})\, e^{-\beta x^{\alpha}}$$

Suppose that Z has a zero truncated power series distribution with the pmf (2). Let $X_{(1)} = \min\{X_1, X_2, ..., X_z\}$ independent of $X\,'s$, then the conditional *pdf* of $X_{(1)} \mid Z$ is obtained as follows

$$f_{X_{(1)}\mid Z}(x\mid z;\alpha,\beta) = z\alpha\beta^2 x^{2\alpha-1} e^{-\beta x^{\alpha}} \left(H(x;\alpha,\beta)\right)^{z-1}.$$

The joint *pdf* of $X_{(1)}$ and Z is as follows

$$f_{X_{(1)}Z}(xz;\alpha,\beta) = \frac{z\alpha\beta^2 a_z \theta^z x^{2\alpha-1} e^{-\beta x^{\alpha}}}{K(\theta)} \left(H(x;\alpha,\beta)\right)^{z-1}.$$

The probability density of a *GMEPS* family of distributions can be defined by the marginal *pdf* of *X*, that is,

$$f(x;\Theta) = \alpha\beta^2\theta x^{2\alpha-1} e^{-\beta x^{\alpha}} \frac{K'\left(\theta H(x)\right)}{K(\theta)}, x,\alpha,\beta,\theta,> 0. \qquad (4)$$

where $\Theta \equiv (\alpha,\beta,\theta)$ is a set of parameters. A random variable *X* with *pdf* (3) is denoted by *X~GMEPS* (α,β,θ).

Furthermore, the *cdf* of *GMEPS* family of distributions corresponding to (3) is obtained as follows

$$F(x;\Theta)=1-\frac{K\left(\theta H(x)\right)}{K(\theta)}. \quad (5)$$

Note that

If $\alpha=1$ the *GMEPS* family is reduced to *MEPS* (Sadaf (2014)).

In addition, the reliability and hazard rate functions for *GMEPS* family of distributions, respectively, take the following forms

$$R(x;\Theta)=\frac{K\left(\theta H(x)\right)}{K(\theta)}, \quad (6)$$

and,

$$h(x;\Theta)=\frac{\alpha\beta^2\theta x^{2\alpha-1}e^{-\beta x^{\alpha}}K'\left(\theta H(x)\right)}{K\left(\theta H(x)\right)}. \quad (7)$$

3. STATISTICAL PROPERTIES OF THE

In this section, some statistical properties including expansion for *pdf* (3), quantile function, rth moment, Re'nyi entropy and distribution of order statistics for the *GMEPS* family of distributions are obtained.

3.1 Useful expansion

In this subsection, two important propositions are provided. The first proposition indicates that the *GMEPS* family *of* distributions has the *GME* distribution as a special limiting case. While the second proposition provides useful expansion for the pdf of *GMEPS* distribution.

Proposition (1)

The *GME* distribution with parameters α *and* β is a limiting special case of *GMEPS* family of distributions when $\theta\to 0^+$.

Proof: By applying $f(\theta)=\sum_{z=1}^{\infty}a_z\theta^z$, for $x>0$ in cdf (4), then we obtain

$$\underset{\theta\to 0^+}{Lim}F(x;\Theta)=1-\underset{\theta\to 0^+}{Lim}\frac{\sum_{z=1}^{\infty}a_z\left(\theta H(x)\right)^z}{\sum_{z=1}^{\infty}a_z\theta^z}.$$

By using L.H. rule, we have

$$\lim_{\theta\to 0^+} F(x;\Theta)=1-\frac{H(x)[1+a_1^{-1}\lim_{\theta\to 0^+}\sum_{z=2}^{\infty} za_z\left(\theta H(x)\right)^{z-1}]}{1+a_1^{-1}\lim_{\theta\to 0^+}\sum_{z=2}^{\infty} za_z\theta^{z-1}}.$$

Hence,

$$\lim_{\theta\to 0^+} F(x;\Theta)=1-(1+\beta x^{\alpha})e^{-\beta x^{\alpha}},$$

which is the *cdf* of the *GME* distribution.

Proposition (2)

The density function of *GMEPS* family can be expressed as a linear combination of the density of $X_{(1)}=\min\{X_1,X_2,...X_z\}$

Proof.

Since $f'(\theta)=\sum_{z=1}^{\infty} za_z\theta^{z-1}$, then the pdf (3) can be expressed as follows

$$f(x;\psi)=\sum_{z=1}^{\infty} P(Z=z;\theta)g_{x_{(1)}}(x;z),$$

where $g_{x_{(1)}}(x;z)$ is the pdf of $X_{(1)}=\min\{X_1,X_2,...,X_z\}$ given by

$$g_{X_{(1)}}(x;z)=z\alpha\beta^2x^{2\alpha-1}(1+\beta x^{\alpha})^{z-1}e^{-z\beta x^{\alpha}},\ x,\alpha,\beta>0.$$

3.2 The Lambert W function

The Lambert W function was developed in 1758 and 1779 by Lambert and Euler respectively. This name Lambert W function, now a days, a standard word in algebra through the solution of equation by computer. In the 1980s, Maple and related material published by Corless et al. (1996) showed almost complete survey this function. This function is based on multivalued which is a solution of the following equation

$$W(z)\exp\left(W(z)\right)=z$$

where z is in general a complex number. The W(z) has two real branches when it becomes real and it is only possible if z is such that $z\geq -1/e$. The symbol W_{-1} is used to denote real negative branch if its values in $(-\infty,-1]$. The symbol W_0 is real positive or principal branch containing values in $[-1,\infty)$.

Lemma 1 Let a,b *and* c be three numbers of complex type, the equation

$z+ab^z=c$ has the solution

$$z=c-\frac{1}{\log(b)}W\left(ab^c\log(b)\right)$$

where W denotes the lambert W function and $z \in C$

3.2.1 Quantile function of the new GMEPS family

In this subsection, the quantile function $Q(p)$ of the *GMEPS* distribution is derived and which is defined by $Q(p) = p$, and is the root of the following equation

$$1 - \frac{K\left(\theta(1+\beta(Q(p))^{\alpha})e^{-\beta(Q(p))^{\alpha}}\right)}{K(\theta)} = p, \ 0 < p < 1.$$

Let $B(p) = -(1+\beta(Q(p))^{\alpha})$. Then,

$$B(p)e^{B(p)} = -\frac{K^{-1}\left((1-p)K(\theta)\right)}{\theta e^{1}}.$$

Then the solution for this $B(p)$ is

$$B(p)e^{B(p)} = W[-\frac{K^{-1}\left((1-p)K(\theta)\right)}{\theta e^{1}}],$$

and where $W(.)$ is the -ve branch of this Lambert W function following to Corless et al. (1996)). Consequently, the $Q(p)$ of the *GMEPS* family is given by solving the following equation for $Q(p)$.

$$(Q(p))^{\alpha} = -\frac{1}{\beta} - W[-\frac{K^{-1}\left((1-p)K(\theta)\right)}{\theta e^{1}}]. \tag{8}$$

3.3 Moments and moment generating function

The rth moment of a r.v X from the *GMEPS* distribution, is

$$\mu_r' = \sum_{z=1}^{\infty} P(Z = z;\theta) \int_0^{\infty} x^r g_{X_{(1)}}(x;z)dx.$$

Then,

$$\mu_r' = \sum_{z=1}^{\infty} P(Z = z;\theta) \int_0^{\infty} z\alpha\beta^2 x^{r+2\alpha-1}(1+\beta x^{\alpha})^{z-1} e^{-z\beta x^{\alpha}} dx.$$

Let $u = \beta x^{\alpha} \rightarrow du = \alpha\beta x^{\alpha-1}dx$, then

$$\mu_r' = \sum_{z=1}^{\infty} zP(Z = z;\theta) \int_0^{\infty} \left(\frac{u}{\beta}\right)^{\frac{r}{\alpha}} u(1+u)^{z-1} e^{-uz} du.$$

By using binomial series more than one times, then

$$\mu_r' = \sum_{z=1}^{\infty} \sum_{i=0}^{z-1} \binom{z-1}{i} z\, P(Z = z;\theta) \int_0^{\infty} \left(\frac{u}{\beta}\right)^{\frac{r}{\alpha}} u^{i} e^{-zu} du.$$

After some simplifications, it takes the following form

$$\mu_r{}' = \sum_{z=1}^{\infty}\sum_{i=0}^{z-1}\binom{z-1}{i}\frac{a_z\theta^z\Gamma\left(\frac{r}{\alpha}+i+1\right)}{K(\theta)z^{\frac{r}{\alpha}+i}\beta^{\frac{r}{\alpha}}}, \quad r=1,2.... \tag{9}$$

Based on the first four moments of the *GMEPS* family, the measures of skewness (SK) and kurtosis (K) can be obtained from following relations respectively

$$SK = \frac{\mu'_3 - 3\mu'_2\,\mu'_1 + 2\mu'^3_1}{(\mu'_2 - \mu'^2_1)^{\frac{3}{2}}}, \qquad K = \frac{\mu'_4 - 4\mu'_3\,\mu'_1 + 6\mu'_2\,\mu'^2_1 - 3\mu'^4_1}{(\mu'_2 - \mu'^2_1)^2},$$

where, $\mu_1{}', \mu_2{}', \mu_3{}'$ and $\mu_4{}'$ can be obtained from (9), by substituting $r = 1,2,3,4.$

Also, *the mgf* $M_X(t)$ is

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r{}',$$

where, $\mu_r{}'$ is the rth raw moment. And then by using (9), the *mgf* of *GMEPS* is as follows:

$$M_X(t) = \sum_{z=1}^{\infty}\sum_{i=0}^{z-1}\binom{z-1}{i}\frac{a_z\theta^z t^r\Gamma\left(\frac{r}{\alpha}+i+1\right)}{r!K(\theta)z^{\frac{r}{\alpha}+i}\beta^{\frac{r}{\alpha}}}, \quad r=1,2....$$

3.4 Order statistics

In this subsection, an expression for the pdf of the ith order statistics from the *GMEPS* distribution is derived. In addition, the distributions of the smallest and largest order statistics are obtained.

Let $X_1, X_2, ...X_n$ be a simple random sample from a *GMEPS* family with pdf (4) and cdf (5). Let $X_{1:n} < X_{2:n} < ... < X_{n:n}$ denote the corresponding order statistics from the sample. The pdf of $X_{i:n}, i = 1,...n$ is given by

$$f_{i:n}(x;\psi) = \frac{1}{\mathrm{B}(i,n-i+1)} f(x;\psi)[F(x;\psi)]^{i-1}[1-F(x;\psi)]^{n-i}, \tag{10}$$

where, $\mathrm{B}(.,.)$ is the beta function. By using cdf (5) and applying the binomial expansion in (10), then we get

$$f_{i:n}(x;\psi) = \frac{f(x;\psi)}{\mathrm{B}(i,n-i+1)}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j\left(\frac{K\left(\theta(1+\beta x^{\alpha})\right)e^{-\beta x^{\alpha}}}{K(\theta)}\right)^{n+j-i}.$$

Now, since an expansion for $\left(K\left(\theta H\left(x\right)\right)\right)^{n+j-i}$ can be written as follows

$$\left(K\left(\theta H\left(x\right)\right)\right)^{n+j-i} = \left(\sum_{z=1}^{\infty}a_z\theta^z e^{-z\beta x^{\alpha}}\left(1+\beta x^{\alpha}\right)^z\right)^{n+j-i},$$

$$\left(K\left(\theta(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}\right)\right)^{n+j-i}=\left(a_1\theta e^{-\beta x^{\alpha}}\left(1+\beta x^{\alpha}\right)\right)^{n+j-i}\times$$
$$\left[1+\frac{a_2}{a_1}\theta e^{-\beta x^{\alpha}}\left(1+\beta x^{\alpha}\right)+\frac{a_3}{a_2}\theta^2 e^{-2\beta x^{\alpha}}\left(1+\beta x^{\alpha}\right)^2+\dots\right]^{n+j-i}.$$

Hence,

$$\left(K\left(\theta(1+\beta x^{\alpha})\right)e^{-\beta x^{\alpha}}\right)^{n+j-i}=a_1^{\,n+j-i}\times$$
$$\left(\sum_{m=0}^{\infty}\ell_m\left(\theta e^{-\beta x^{\alpha}}\left(1+\beta x^{\alpha}\right)^m\right)\right)^{n+j-i},\ell_m=\frac{a_{m+1}}{a_1},\ m=1,2,\dots \quad (11)$$

According to Gradshteyn and Ryzhik (2000) for a positive integer, we have the following relation

$$\left(\sum_{m=0}^{\infty}\ell_m Y^m\right)^{n+j-i}=\sum_{m=0}^{\infty}d_{n+j-i,m}Y^m.$$

Then (11) can be written as follows

$$\left(K\left(\theta(1+\beta x^{\alpha})\right)e^{-\beta x^{\alpha}}\right)^{n+j-i}=(a_1)^{n+j-i}\sum_{m=0}^{\infty}d_{n+j-i,m}\left(\theta\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right)^{n+j-i+m}, \quad (12)$$

where, $d_{n+j-i,0}=1$ and the coefficients $d_{n+j-i,m}$ are easily determined from the following recurrence equation

$$d_{n+j-i,t}=t^{-1}\sum_{m=1}^{t}[m(n+j-i+1)-t]\ell_m d_{n+j-i,t-m},t\geq 1.$$

In addition,

$$K'\left(\theta(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}\right)=\sum_{z=1}^{\infty}z\,a_z\left(\theta(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}\right)^{z-1}.$$

Let $k=z-1$, then the previous equation can be expressed as

$$K'\left(\theta(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}\right)=\sum_{k=0}^{\infty}\ell_k(k+1)\left(\theta(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}\right)^k,\ \ell_k=\frac{a_{k+1}}{a_1} \quad (13)$$

Then, the pdf of the ith order statistic from *GMEPS* family of distributions is obtained by substituting expansions (12) and (13) in pdf (10) as follows

$$f_{i:n}(x;\Theta)=\frac{\beta^2\alpha\theta x^{2\alpha-1}e^{-\beta x^{\alpha}}\sum_{k=0}^{\infty}\ell_k(k+1)\left(\theta(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}\right)^k}{\mathrm{B}(i,n-i+j)(K(\theta))^{n+j-i+1}}$$
$$\times\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j a_1^{\,n+j-i+1}\sum_{m=0}^{\infty}d_{n+j-i,m}\left(\theta(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}\right)^{n+j-i+m}.$$

Thus, the pdf of the ith order statistics can be formed as follows

$$f_{i:n}(x;\Theta)=\frac{\beta^{2}\alpha x^{2\alpha-1}}{\mathrm{B}(i,n-i+j)}\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{`m=0}^{\infty}(-1)^{j}\binom{i-1}{j}\ell_{k}(k+1)$$
$$\times\frac{d_{n+j-i,m}a_{1}^{n+j-i+1}\theta^{n+j-i+m+k+1}e^{-(n+j-i+m+1+k)\beta x^{\alpha}}}{(K(\theta))^{n+j-i+1}}\left(1+\beta x^{\alpha}\right)^{n+j-i+m+k},\ \ x>0.$$

or

$$f_{i:n}(x;\Theta)=\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{`m=0}^{\infty}\tau_{j,k,m}\beta x^{2\alpha-1}\left(1+\beta x^{\alpha}\right)^{n+j-i+m+k}e^{-(n+j-i+m+k+1)\beta x^{\alpha}},\quad \text{where,}$$

$$\tau_{j,k,m}=(-1)^{j}\binom{i-1}{j}\frac{\alpha\lambda\ell_{k}(k+1)\theta^{n+j-i+m+k+1}a_{1}^{n+j-i+1}d_{n+j-i,m}}{\mathrm{B}(i,n-i+j)(K(\theta))^{n+j-i+1}}.$$

Another form can be written by using binomial expansion as follows:

$$f_{i:n}(x;\psi)=\beta\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{`m=0}^{\infty}\sum_{h=0}^{n+j-i+m+k}\eta_{j,k,m,h}x^{\alpha(h+1)}e^{-(n+j-i+m+k+1)\beta x^{\alpha}},\qquad(14)$$

where,

$$\eta_{j,k,m,h}=(-1)^{j}\binom{i-1}{j}\binom{m+n+j-i+k}{h}\frac{\alpha\beta^{h+1}\theta^{n+j-i+m+k+1}\ell_{k}(k+1)a_{1}^{n+j-i+1}d_{n+j-i,m}}{\mathrm{B}(i,n-i+j)(K(\theta))^{n+j-i+1}}.$$

In particular, the pdf of the smallest and the largest order statistics of the *GMEPS* distribution is obtained by substituting $i=1,n$, in (14), respectively, as follows

$$f_{1:n}(x;\psi)=\sum_{k=0}^{\infty}\sum_{`m=0}^{\infty}\sum_{h=0}^{n+j-i+m+k}\phi_{k,m,h}\beta x^{\alpha(h+1)}e^{-(n+m+k)\beta x^{\alpha}},$$

$$\phi_{k,m,h}=\binom{m+n+-1+k}{h}\frac{n\alpha\beta^{h+1}\ell_{k}(k+1)\theta^{n+m+k}a_{1}^{n}d_{n-1,m}}{(K(\theta))^{n}}.$$

and,

$$f_{n:n}(x;\psi)=\sum_{k=0}^{\infty}\sum_{j=0}^{n-1}\sum_{`m=0}^{\infty}\sum_{h=0}^{j+m+k}\varsigma_{j,k,m,h}\beta x^{\alpha(h+1)}e^{-(j+m+k+1)\beta x^{\alpha}},$$

where,

$$\varsigma_{k,m,h}=\binom{m+j+k}{h}\binom{n-1}{j}(-1)^{j}\frac{n\beta^{h+1}\alpha\ell_{k}(k+1)\theta^{j+m+k+1}a_{1}^{j+1}d_{j,m}}{(K(\theta))^{j+1}}.$$

3.5 ~~Re'nyi~~ Rényi Entropy $I_R(x)$

In engineering and science various situations where entropy is used. The entropy of an *r.v* X is a measure of variation of the uncertainty. If X is an *r.v* distributed to *GMEPS*, then $I_R(x)$, for $\rho>0,$ and $\rho\neq 1,$ is defined as

$$I_R(x) = (1-\rho)^{-1}\log_b\left(\int_0^\infty \left(f\ (x;\psi)\right)^\rho dx\right).$$

Let, $IP = \int_0^\infty \left(f\ (x;\psi)\right)^\rho dx$, then IP can be written as follows:

$$IP = \int_0^\infty \left(\alpha\beta^2\theta x^{2\alpha-1}e^{-\beta x^\alpha}\right)^\rho \left\{\frac{\sum_{z=1}^\infty z a_z \left(\theta(1+\beta x^\alpha)\ e^{-\beta x^\alpha}\right)^{z-1}}{K(\theta)}\right\}^\rho dx.$$

But

$$\left(\sum_{z=1}^\infty z a_z \left(\theta(1+\beta x^\alpha)\ e^{-\beta x^\alpha}\right)^{z-1}\right)^\rho = a_1^\rho\left(\sum_{m=0}^\infty \delta_m \left(\theta(1+\beta x^\alpha)\ e^{-\beta x^\alpha}\right)^m\right)^\rho, \delta_m = \frac{a_{m+1}}{a_1},\ m = 1, 2, \ldots$$

Using the same rule as provided by Gradshteyn and Ryzhik (2000), then we obtain

$$\left(\sum_{z=1}^\infty \delta_m \left(\theta(1+\beta x^\alpha)\ e^{-\beta x^\alpha}\right)^m\right)^\rho = \sum_{m=0}^\infty d_{\rho,m}\left(\theta(1+\beta x^\alpha)\ e^{-\beta x^\alpha}\right)^m.$$

Therefore,

$$\left(\sum_{z=1}^\infty z a_z \left(\theta(1+\beta x^\alpha)\ e^{-\beta x^\alpha}\right)^{z-1}\right)^\rho = a_1^\rho \sum_{z=1}^\infty d_{\rho,m}\left(\theta(1+\beta x^\alpha)\ e^{-\beta x^\alpha}\right)^m. \qquad (15)$$

The coefficients for $t > 1$ are computed from the following recurrence equation:

$$d_{\rho,t} = t^{-1}\sum_{m=1}^t [m(\rho+1)-t]\delta_m d_{\rho,t-m}, d_{\rho,0} = 1$$

Using binomial expansion for $\left(1+\lambda x^\alpha\right)^m$, then (15) will be as follows:

$$\left(\sum_{z=1}^\infty z a_z \left(\theta(1+\beta x^\alpha)\ e^{-\beta x^\alpha}\right)^{z-1}\right)^\rho = a_1^\rho \sum_{z=1}^\infty \sum_{k=0}^m \binom{m}{k} d_{\rho,m}\theta^m e^{-m\beta x^\alpha}\left(\beta x^\alpha\right)^k$$

Then the IP can be rewritten as follows

$$IP = \int_0^\infty \left(\alpha\beta\theta x^{\alpha-1}a_1\right)^\rho (1+\beta x^\alpha)^\rho \sum_{m=0}^\infty \sum_{k=0}^m d_{\rho,m}\theta^m \binom{m}{k} (\beta x^\alpha)^k e^{-(m+\rho)\beta x^\alpha} dx,$$

$$= \sum_{m=0}^\infty \sum_{k=0}^m \sum_{h=0}^\rho \binom{m}{k}\binom{\rho}{h} d_{\rho,m}\theta^m \int_0^\infty \left(\alpha\beta\theta x^{\alpha-1}a_1\right)^\rho (\beta x^\alpha)^{k+h} e^{-(m+\rho)\beta x^\alpha} dx.$$

After some simplification, then the Re'nyi entropy takes the following form

$$I_R(x) = (1-\rho)^{-1}\log_b\left[\sum_{m=0}^\infty \sum_{k=0}^m \sum_{h=0}^\rho \binom{m}{k}\binom{\rho}{h} \frac{d_{\rho,m}\theta^{m+\rho}\alpha^{\rho-1}a_1^\rho \Gamma\left(\frac{\rho(\alpha-1)+1}{\alpha}+k+h\right)}{(K(\theta))^\rho (m+\rho)^{\frac{\rho(\alpha-1)+1}{\alpha}+k+h}}\right]. \qquad (16)$$

4. Special models of the *GMEPS* family

Some sub-models from *GMEPS* family of distributions for selected values of the parameters are presented in this section. Also, some sub-models; which are the generalized moment exponential Poisson and moment exponential Poisson distributions are discussed in more details.

The sub models are considered as follows:

1. For $K(\theta)=e^{\theta}-1$, then the *GMEPS* distribution reduces to generalized moment exponential Poisson *(GMEP)* distribution with the following cdf:

$$F(x;\psi)=\frac{e^{\theta}-\exp\left[\theta\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right]}{e^{\theta}-1},\qquad x,\alpha,\lambda,\beta>0. \tag{17}$$

2. For $K(\theta)=e^{\theta}-1, \alpha=1,$ then the *GMEPS* distribution reduces to moment exponential Poisson (*MEP*) distribution with the following cdf:

$$F(x;\beta,\theta)=\frac{e^{\theta}-\exp\left[\theta\left(1+\beta x\right)e^{-\beta x}\right]}{e^{\theta}-1},\qquad x,\beta,\theta>0.$$

3. For $K(\theta)=-\ln(1-\theta)$ then the *GMEPS* distribution reduces to generalized moment exponential logarithmic (*GMEL*) distribution with the following cdf:

$$F(x;\psi)=1-\frac{\ln\left[1-\theta\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right]}{\ln(1-\theta)},\qquad x,\beta,\alpha>0,\quad 0<\theta<1.$$

$$f(x)=\frac{\theta\left(2+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\alpha\beta x^{\alpha-1}}{\ln(1-\theta)\left(1-\theta\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right)}$$

4. For $K(\theta)=-\ln(1-\theta), \alpha=1,$ then the *GMEPS* distribution reduces to moment exponential logarithmic (*MEL*) distribution with the following cdf:

$$F(x;\theta,\beta)=1-\frac{\ln\left[1-\theta\left(1+\beta x\right)e^{-\beta x}\right]}{\ln(1-\theta)},\qquad x>0,\quad 0<\theta<1.$$

5. For $K(\theta)=\theta(1-\theta)^{-1}$, then the *GMEPS* distribution reduces to generalized moment exponential geometric (*MEG*) distribution with the following cdf:

$$F(x;\psi)=\frac{1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}}{1-\theta\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}},\qquad x,\beta,\alpha>0,\quad 0<\theta<1. \tag{18}$$

6. For $K(\theta)=\theta(1-\theta)^{-1}, \alpha=1$ then the *GMEPS* distribution reduces to moment exponential geometric (*MEG*) distribution with the following cdf:

$$F(x;\lambda,\theta)=\frac{1-\left(1+\beta x\right)e^{-\beta x}}{1-\theta\left(1+\beta x\right)e^{-\beta x}},\qquad x,\beta>0,\ 0<\theta<1.$$

7. For $K(\theta)=(1-\theta)^{m}-1,$ then the *GMEPS* distribution reduces to generalized

moment exponential binomial (*GMEB*) distribution with the following cdf:

$$F(x;\psi)=\frac{(1-\theta)^m-\left[1-\theta\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right]^m}{(1-\theta)^m-1},\qquad x,\beta,\alpha>0,\quad 0<\theta<1.$$

4.1 Generalized moment exponential Poisson distribution

As mentioned above the *GMEP* distribution is obtained from *GMEPS* family distribution as a special case. The pdf of the *GMEP* distribution corresponding to (17) takes the following form

$$f(x;\psi)=\frac{\alpha\beta^2\theta x^{2\alpha-1}e^{-\beta x^{\alpha}}\exp\left(\theta(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}\right)}{(e^{\theta}-1)}\quad ,x,\beta,\alpha,\theta>0. \tag{19}$$

In addition, the reliability and hazard rate function take the following form respectively:

$$R(x;\psi)=\frac{\exp\left[\theta\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right]-1}{e^{\theta}-1},$$

and, $$h(x;\psi)=\frac{\alpha\beta^2\theta x^{2\alpha-1}e^{-\beta x^{\alpha}}\exp\left(\theta(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}\right)}{\left[\exp\left(\theta(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}\right)-1\right]}.$$

Figure 1, gives plots of the pdf of the *GMEP* distribution for some parameters values exhibiting the behavior of density.

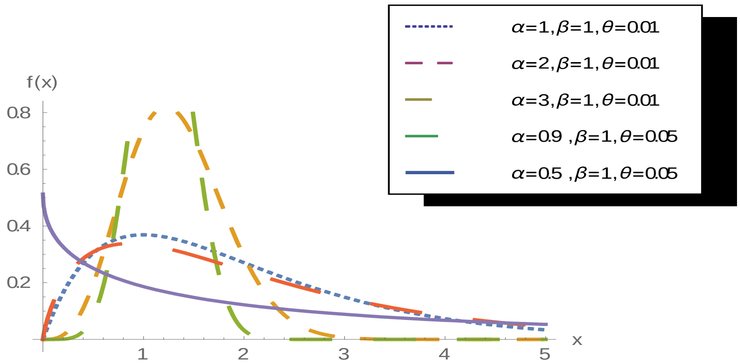

Figure 1.The pdf plots of the *GMEP* distribution

The following figure gives the hazard rate function plots for *GMEP* distribution for some parameters values.

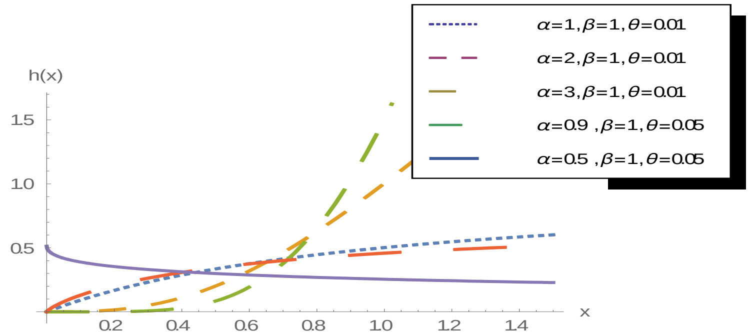

Figure 2. The hazard rate plots for the *GMEP* distribution

It is clear from Figure 2that the *GMEP* distribution has increasing, decreasing and constant failure rates.

The quantile function for the *GMEP* distribution is obtained directly from expression (8) with $K(\theta)=e^{\theta}-1$, and $K^{-1}(\theta)=\ln(1+\theta)$ as follows:

$$(Q(p))^{\alpha}=-\frac{1}{\lambda}-W[-\frac{\ln(p+(1-p)e^{\theta})}{\theta e^{1}}].$$

Solving this equation for $Q(p)$, the quantile function of *GMEP* is obtained.

Furthermore, the rth moment about zero for the *GMEP* distribution is given by substituting the following pmf of truncated Poisson

$$P(Z=z;\theta)=\frac{e^{-\theta}\theta^{z}}{z!(1-e^{-\theta})},\ z=1,2,...$$

in (9) as follows

$$\mu_r{}'=\sum_{z=1}^{\infty}\sum_{j=0}^{z-1}\sum_{i=0}^{j+1}\binom{z-1}{j}\binom{j+1}{i}\frac{\theta^{z}\Gamma\left(\frac{r}{\alpha}+i+1\right)}{z!(e^{\theta}-1)z^{\frac{r}{\alpha}+i}\lambda^{\frac{r}{\alpha}}},$$

$$r=1,2....$$

Additionally the Re'nyi entropy is obtained by substituting $K(\theta)=e^{\theta}-1$, in (16) as follows

$$I_R(x)=(1-\rho)^{-1}\log_b\left[\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{h=0}^{\rho}\binom{m}{k}\binom{\rho}{h}\frac{d_{\rho,m}\theta^{m+\rho}\alpha^{\rho-1}a_1^{\rho}\Gamma(\frac{\rho(\alpha-1)+1}{\alpha}+k+h)}{(e^{\theta}-1)^{\rho}(m+\rho)^{\frac{\rho(\alpha-1)+1}{\alpha}+k+h}}\right].$$

4.2 Generalized moment exponential geometric distribution

The generalized moment exponential geometric distribution is discussed as the second special model from *GMEPS* family. The pdf of the *GMEG* distribution corresponding to (18) takes the following form

$$f(x;\psi)=\frac{\alpha\beta^{2}x^{2\alpha-1}e^{-\beta x^{\alpha}}(1-\theta)}{\left[1-\left(\theta\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right)\right]^{2}},\qquad x>0,0<\theta<1,\alpha,\beta>0. \tag{20}$$

In addition, the reliability and hazard rate function take the following form:

$$R(x;\psi)=\frac{(1-\theta)\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}}{1-\theta\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}},$$

and,

$$h(x;\psi)=\frac{\alpha\beta^{2}x^{2\alpha-1}}{\left(1+\beta x^{\alpha}\right)\left[1-\left(\theta\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right)\right]}.$$

Figures 3 and 4 represent *pdf* and *hrfs* plots for *GMEG* distribution for some selected values of parameters.

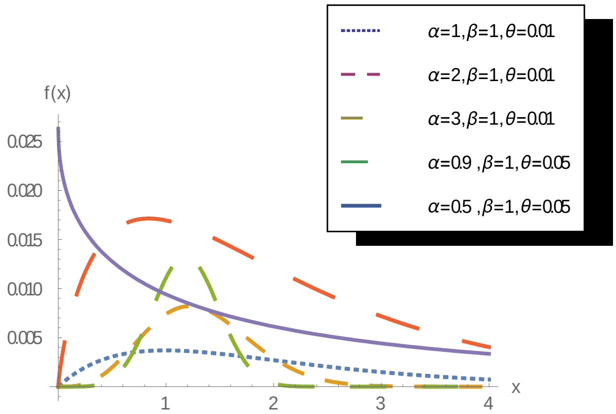

Figure.3. The pdf plots of the *GMEG* distribution

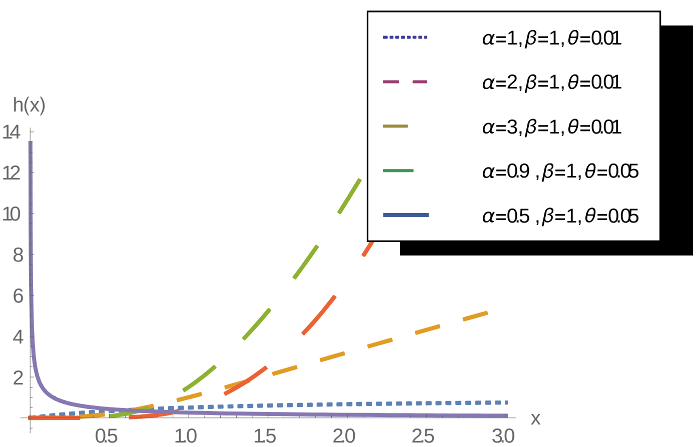

Figure. 4. The hazard rate plots of the *GMEG* distribution

From this figure, it is observed that the shapes of the *hrfs* are increasing at some parameter values. For some choices of parameters; the distribution has increasing, decreasing and constant patterns.

The quantile function for the *GMEG* distribution is obtained directly from expression (8) with $K(\theta)=\theta(1-\theta)^{-1}$, and $K^{-1}(\theta)=\theta(1+\theta)^{-1}$ as follows

$$(Q(p))^{\alpha}=-\frac{1}{\lambda}-W[-\frac{(1-p)}{(1-\theta p)e^{1}}].$$

Solving this equation for $Q(p),$ the quantile function *GMEG* is obtained.

Additionally, the rth moment about zero for the *GMEG* distribution is given by substituting the following pmf of truncated geometric $P(Z=z;\theta)=(1-\theta)\theta^{z-1},\ z=1,2,...,$ in (9) as follows

$$\mu_r{}'=\sum_{z=1}^{\infty}\sum_{j=0}^{z-1}\sum_{i=0}^{j+1}\binom{z-1}{j}\binom{j+1}{i}\frac{\theta^{z-1}(1-\theta)\Gamma\left(\frac{r}{\alpha}+i+1\right)}{z^{\frac{r}{\alpha}+i}\lambda^{\frac{r}{\alpha}}},\ r=1,2.... \qquad (21)$$

Further, the Re'nyi entropy is obtained by substituting $C(\theta)=\theta(1-\theta)^{-1},$ in (16) as follows

$$I_R(x)=(1-\rho)^{-1}\log_b\left[\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{h=0}^{\rho}\binom{m}{k}\binom{\rho}{h}\frac{d_{\rho,m}\theta^{m}\lambda^{\rho+h+k}\alpha^{\rho-1}a_1^{\rho}\Gamma(\frac{\rho(\alpha-1)+1}{\alpha}+k+h)}{(1-\theta)^{-\rho}(m+\rho)^{\frac{\rho(\alpha-1)+1}{\alpha}+k+h}}\right].$$

5. Parameter estimation of the *GMEPS* family

In this section, parameters' estimation of *GMEPS* family of distributions is obtained by using the maximum likelihood method.

Let $X_1,X_2,...X_n$ be a simple random sample from the *GMEPS* family with set of parameters $\psi\equiv(\alpha,\beta,\theta).$ The log-likelihood function based on the observed random sample of size n is given by:

$$f(x;\psi)=\alpha\,\beta^{2}\theta\,x^{2\alpha-1}e^{-\beta x^{\alpha}}\frac{K'\left(\theta(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}\right)}{K(\theta)},x,\beta,\alpha,\theta,>0.$$

$$L(x;\psi)=\alpha^{n}\beta^{2n}\left(\prod_{i=1}^{n}x\right)^{2\alpha-1}e^{-\beta\sum_{i=1}^{n}x^{\alpha}}\frac{\prod_{i=1}^{n}K'\left(\theta(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}\right)}{\left(K(\theta)\right)^{n}}$$

$$\ln L(x;\psi) = n\ln\alpha + 2n\ln\beta + \left(2\ \alpha - 1\right)\sum_{i=1}^{n} x_i - \beta\ \sum_{i=1}^{n} x_i^{\ \alpha} + \sum_{i=1}^{n}\ln\left(K'(\theta S_i)\right) - n\ln(K(\theta)).$$

where, $\ln L = \ln L(x;\psi)$ and $S_i = (1+\beta x_i^{\alpha})e^{-\beta x_i^{\alpha}}$.

The partial derivatives of the log-likelihood function with respect to the unknown parameters are given by:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - \beta\sum_{i=1}^{n} x_i^{\alpha}\ln x_i + 2\sum_{i=1}^{n}\ln x_i - \theta\sum_{i=1}^{n}\frac{K''(\theta S_i)}{K'(\theta S_i)}\frac{\partial S_i}{\partial \alpha},$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} x_i^{\alpha} + \theta\sum_{i=1}^{n}\frac{K''(\theta S_i)}{K'(\theta S_i)}\frac{\partial S_i}{\partial \beta},$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\left[\frac{K''(\theta S_i)}{K'(\theta S_i)}\right]S_i - \frac{nK'(\theta)}{K(\theta)},$$

where,

$$\frac{\partial S_i}{\partial \alpha} = -\beta^2 x_i^{2\alpha} e^{-\beta x_i^{\alpha}}\ln x_i,$$

and,

$$\frac{\partial S_i}{\partial \beta} = -\lambda x_i^{2\alpha}.$$

The ML estimates of the model parameters can be found by solving the non-linear equations $\frac{\ln L}{\partial \alpha} = 0, \frac{\ln L}{\partial \beta} = 0, \frac{\ln L}{\partial \theta} = 0.$ These equations can be solved numerically and an iterative technique may be used through statistical software.

5.1. A Simulation Studies:

We adopt the Monte Carlo simulation study to access performance of ML estimator's of $\Theta = (\alpha, \beta, \theta)$ through Mathematica 10.2 version. We generate different *n* sample observation from the quantile function in equation (20) above of the model *GMEG* distribution. The parameters are estimated by ML method. We considered different sample size =30, 50, 100, 300, 500 and 1000 and the number of repetition is 10000. The true values of α, β and θ with three different sets of values, in table 1 of below shows the bias with corresponding mean squared error (*MSE*) of the estimated parameters. We observed that the bias and Mean square error for the *GMEG* model given below as: (α, β, θ) decreases.

Table 1. The Bias and MSE on Monte Carlo simulation for parameters values for the *GMEG* model

Parameter	True value	Sample size n	Mean	Bias	MSE
α	2	$n=30$	2.2437	0.2437	1.0321
		$n=50$	2.2321	0.2321	0.9014
		$n=100$	2.2232	0.2232	0.7932
		$n=300$	2.1524	0.1524	0.5012
		$n=500$	2.0517	0.0517	0.3223
		$n=1000$	2.0039	0.0039	0.2015
β	3	$n=30$	3.2537	0.2537	0.9423
		$n=50$	3.2420	0.2420	0.8317
		$n=100$	3.2412	0.2412	0.7694
		$n=300$	3.2015	0.2015	0.7062
		$n=500$	3.1436	0.1436	0.4319
		$n=1000$	3.0219	0.0219	0.1726
θ	0.5	$n=30$	0.6813	0.1813	0.4536
		$n=50$	0.6801	0.1801	0.3998
		$n=100$	0.6521	0.1521	0.3457
		$n=300$	0.5523	0.0523	0.1929
		$n=500$	0.5176	0.0176	0.1612
		$n=1000$	0.5069	0.0069	0.0134

Given first three sample moments, the corresponding $\Theta=(\alpha,\beta,\theta)$ values are estimated from the actual theoretical first three population moments derived from (The sampling distributions of estimated $\Theta=(\alpha,\beta,\theta)$ are given in Table 3 based on various sample sizes. For small samples, the percentage of estimates falling in the indicated interval increases with larger sample size. Using this range, we estimate Θ by the method of moments. If we include omitted data, we expect larger Mean Square Error (MSE). This MSE, however, decreases with increasing sample size.

Table 2: Percentage of sample estimates of $\Theta=(\alpha,\beta,\theta)$ through method of moments (MM) for the *GMEG* model

n	% estimated values of parameter in indicated interval with $\alpha=2$	% estimated values of parameter in indicated interval with $\beta=3$	% estimated values of parameter in indicated interval with $\theta=0.5$

	$1.4 < \hat{\alpha} < 2.6$	$2.5 < \hat{\beta} < 3.5$	$0.3 < \hat{\theta} < 0.7$
30	87.58%	86.18%	80.02%
50	93.04%	90.26%	85.52%
100	97.35%	93.94%	88.71%
250	98.92%	97.42%	94.56%
500	99.59%	99.01%	96.69%
1000	99.86%	99.45%	98.94%

6. APPLICATIONS

In this section, the flexibility of some special models of *GMEPS* family is examined using two real data sets. We illustrate the superiority of new selected distribution as compared with some sub-models.

6.1 Aircraft Windshield data set

The first data set correspond the failure times of 84 for a particular model aircraft windshield. This data are reported in the book "Weibull Models" by Murthy et al.(2004, p.297)[12]. This data consist of 84 failed windshield, the unit for measurement is 1000 h. The data are :0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309,1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070,1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281,2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432,2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506,2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619,2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757,2.324, 3.376, 4.663.

We estimated unknown parameters of the distribution by maximum likelihood method as describe in section 5 by using the R code to find the best fit of the data. We use some measures of goodness of fit, including Kolmogorov Smirnov (K-S),
For this real data set, we have fitted generalized moment exponential geometric, Weibull distribution, exponentiated exponential distribution and exponential distribution.

Table 3. Criteria for comparison for second data set

Model	$k-s$	*AIC*	*CAIC*	*BIC*
GMEG	0.681	263.58	195.89	268.96
WD	0.742	264.10	205.06	270.87
EE	0.721	283.68	227.93	288.54
E	0.694	327.75	218.85	330.18

Smaller values of these statistics indicate a better fit. Tables 3 and 4 compare the *GMEG* distribution with the WD, EE, and E. Moreover, values of K-S, AIC, AICC, and BIC, are listed in Tables 4. According to the criterion K-S, AIC, AICC, and BIC, we found that GMEG distribution is the best fitted model than the models WD, EE, and E distributions for the Aarset data set and for the aircraft windshield data set. So, the GMEG model could be chosen as the best model. The histogram of two data sets and the estimated PDFs, CDFs and P-P plots for the fitted data model are displayed in Figures (5, 6, 7, 8, 9, 10). It is clear from Tables 4 and Figures (5, 6, 7, 8, 9, 10) that the GMEG provides a better fit to the histogram and therefore could be chosen as the best model for both data set. Also the plots of the estimated densities and estimated cumulative of the fitted models are achieved in Figures 5 and 6.

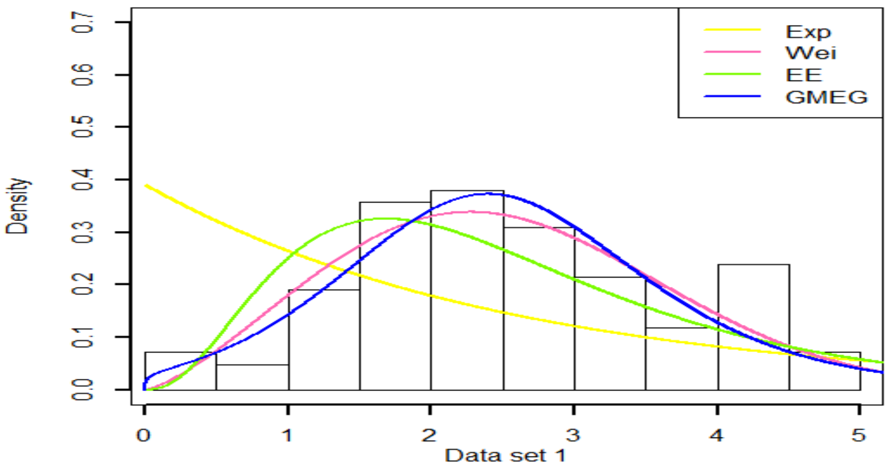

Figure 5. Estimated densities of models for the second data set

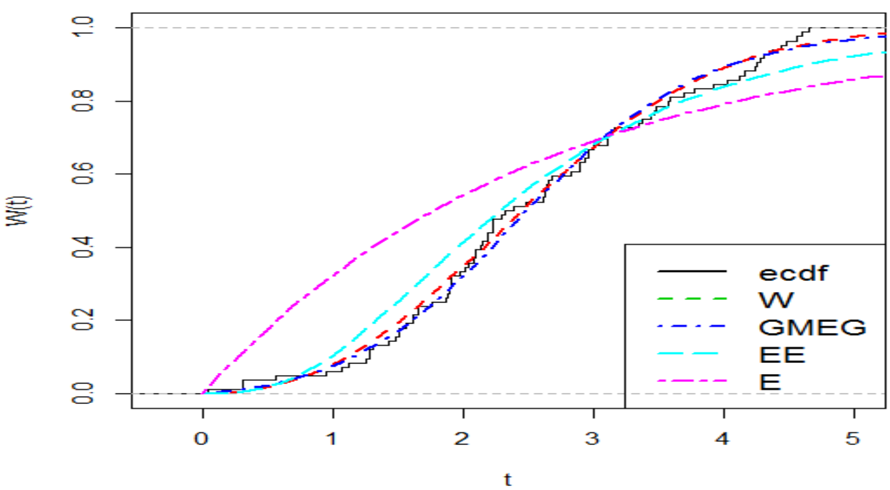

Figure 6 Estimated cumulative densities of models for the first data set

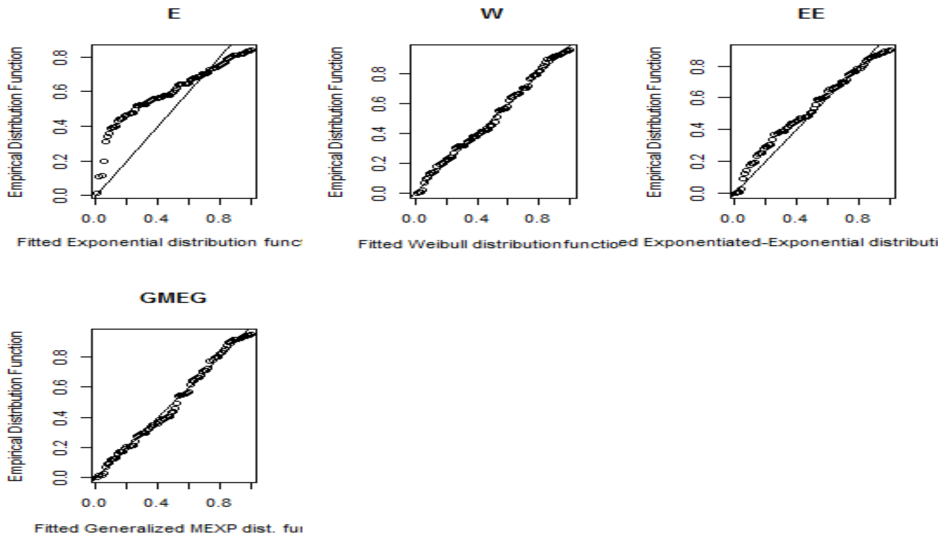

Figure 7: The probability–probability plots for the aircraft windshield data set

6.2 ~~2nd data set~~

The second data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The data are as follows:

0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55, 2.54, 0.77.

Table 4. Criteria for comparison for 2nd data set

Model				
	$k-s$	*AIC*	*CAIC*	*BIC*
GMEG	**0.823**	**193.53**	**193.87**	**200.34**
WD	**0.832**	**196.06**	**196.22**	**200.60**
EE	**0.853**	**194.95**	**195.33**	**201.50**
E	**0.844**	**226.89**	**226.95**	**229.16**

For the second data set, the values of *k-s*, *AIC*, *BIC* and *CAIC* are record in table 4

The plots of the estimated cumulative and estimated densities of the fitted models are achieved in Figures. 8 and 9 respectively.

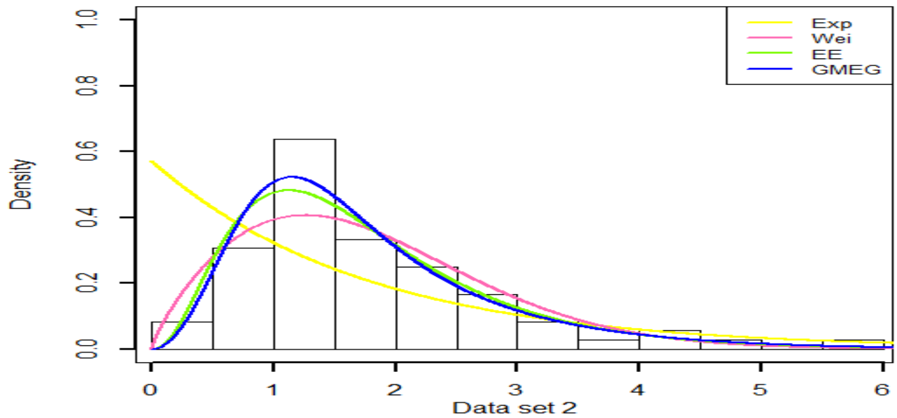

Figure 8. Estimated densities of models for the Bjerkedal (1960) data set

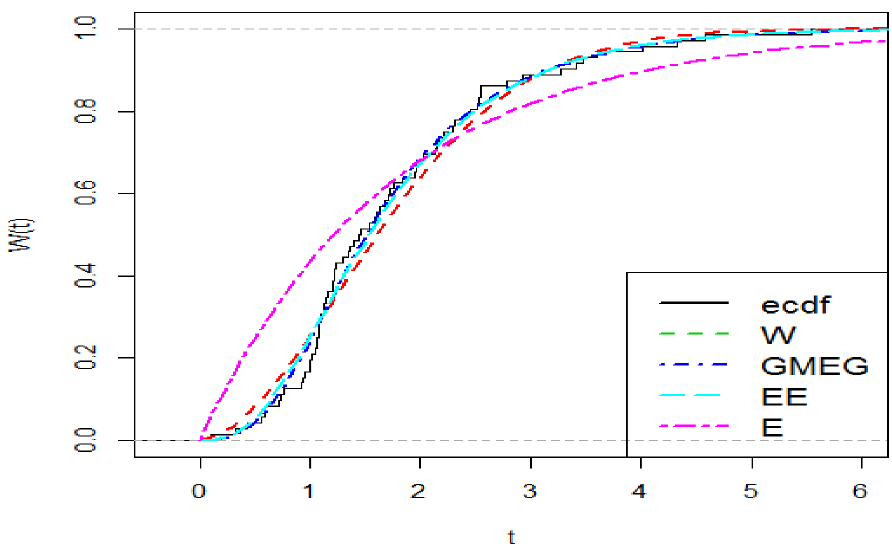

Figure. 9. Estimated cumulative densities of models for the second data set

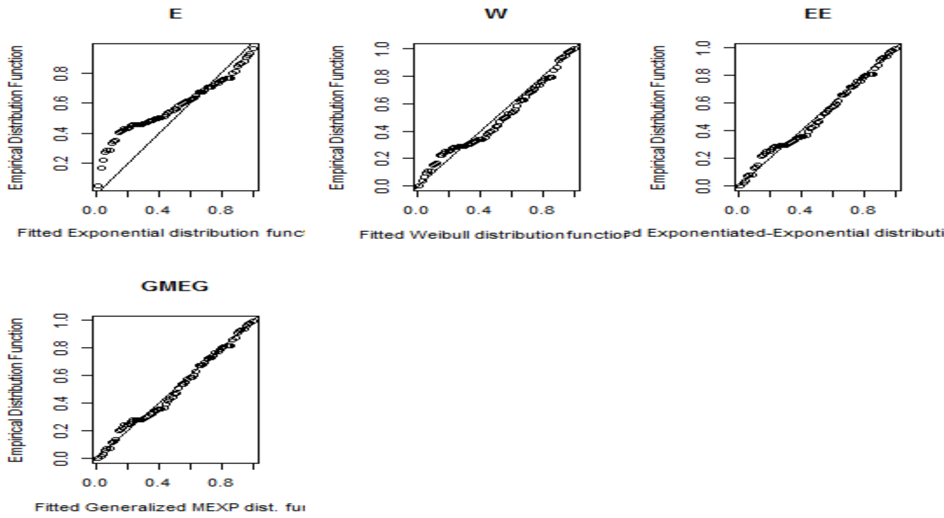

Figure 10: The probability–probability plots for the Bjerkedal (1960) data set

It is clear from the above two figures that the new model *GMEG* has the best fit in the class of its competitor distributions.

7. Conclusion

We introduce a new class of lifetime models called the generalized moment exponential power series**.** This new family is obtained by compounding the generalized moment exponential distribution and truncated power series distributions. More specifically, the generalized moment exponential power series covers several new distributions. Also, mathematical properties of the new family, including expressions for density function, moments, moment generating function, quantile function, order statistics and entropy are provided. The hazard function has various shapes such as increasing, decreasing, and bathtub. By simulation procedures it is discovered that the ML estimators are consistent since the bias and MSE approach to zero when the sample size increases. The usefulness of the model associated with this family is illustrated by two real data sets and the new model provides a better fit than the models provided in literature.

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