# Oscillation criteria for a class of third-order differential equations with neutral term 


#### Abstract

This paper is concerned with oscillation criteria for a class of third-order differential equations with neutral term by using some necessary analysis techniques, some sufficient conditions for oscillation are obtained, some examples are provided to illustrate the main results.


Keywords: Oscillation; Neutral; Third-order; Differential equations
2010 AMS Subject Classification: 41A25, 65D99

## 1 Introduction

In this paper, we consider the oscillatory and asymptotic properties for a class of thirdorder nonlinear differential equation with damped term

$$
\begin{equation*}
\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}[x(t)+a(t) x(\mu(t))]^{\prime}\right)^{\prime}\right)^{\prime}+q(t) f(x(\delta(t)))=0, \quad t \geq t_{0} \tag{E}
\end{equation*}
$$

As usual, we use the notation, $u(t)=x(t)+a(t) x(\mu(t))$. In what follows, it is always assume
(C1) $p(t), r(t), a(t), q(t), \delta(t), \mu(t) \in C\left(\left[t_{0}, \infty\right),(0, \infty)\right)$,
(C2) $\int_{t_{0}}^{\infty} p(t) \mathrm{d} t=\int_{t_{0}}^{\infty} r(t) \mathrm{d} t=\infty, r^{\prime}(t)>0$,
(C3) $\mu(t) \leq t, \lim _{t \rightarrow \infty} \mu(t)=\lim _{t \rightarrow \infty} \delta(t)=\infty$,
(C4) $0 \leq a(t) \leq a_{0}<1, f \in C(R, R), f^{\prime}(v)>0, \frac{f(v)}{v} \geq \lambda$, for all $v \neq 0$, and for some $\lambda>0$.

By a solution of equation (E) we mean a continuous function $x(t)$ definned on an interval $\left[t_{0}, \infty\right)$ such that $\left(\frac{1}{r(t)}[x(t)+a(t) x(\mu(t))]^{\prime}\right)^{\prime}$ is continuously differentiable satifies (E), we assum that equation (E) have such solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $\left[t_{0}, \infty\right)$, otherwise, it is called nonoscillatory. We say equation ( E ) is oscillatory if all its continuable solutions are oscillatory.

In what follows, we consider only proper solution of the equation (E) which are defined for all large $t$. More and more people are interested in oscillatory and nonoscillatory criteria to be shown ${ }^{[1-7]}$. Our principal goal in this paper is to derive new oscillation criteria for equation (E), without requiring restrictive condition (4) and (7) in [1]

For simplicity, we introduce the following nonation:

$$
u^{[0]}(t)=u(t), \quad u^{[1]}(t)=\frac{1}{r(t)} u^{\prime}(t), \quad u^{[2]}(t)=\frac{1}{p(t)}\left(u^{[1]}(t)\right)^{\prime}
$$

lemma 1. Let $x(t)$ be a nonscillatory solutionof (E), then there exists a $T_{x}$ for $t>$ $T_{x} \geq t_{0}$, such that $u(t)$ has only the following two cases.
(i) $u(t) u^{[1]}(t)<0, \quad u(t) u^{[2]}(t)>0$,
(ii) $u(t) u^{[1]}(t)>0, u(t) u^{[2]}(t)>0$.

Proof. Without loss of generality we may assume that $x(t)$ is eventually positive, i.e. there exists $T_{x} \geq t_{0}$ such that $x(t)>0, u(t)>0$ for $t \geq T_{x}$. (If it is an eventually negative ,the proof is similar). Using (E) we get $\left(u^{[2]}(t)\right)^{\prime}<0$, eventually. Then $u^{[2]}(t)$ is decreasing and of one sign for $t \geq T_{x}$. If we admit $u^{[2]}(t)<0$, then there exists aconstant $M>0$ such that

$$
\begin{gathered}
\left.\frac{1}{p(t)} u^{[1]}(t)\right)^{\prime} \leq-M<0, \\
\left(u^{[1]}(t)\right)^{\prime} \leq-M p(t)
\end{gathered}
$$

Integrating from $T_{x}$ to $t$, we obtain

$$
u^{[1]}(t) \leq u^{[1]}\left(T_{x}\right)-M \int_{T_{x}}^{t} p(s) \mathrm{d} s
$$

Leting $t \rightarrow \infty$ and using (C2), we get $u^{[1]}(t)<0$, which together with $r^{\prime}(t)>0$ and $u^{[2]}(t)=\frac{r(t) u^{\prime \prime}(t)-r^{\prime}(t) u^{\prime}(t)}{p(t) r^{2}(t)}<0$.

We get $u^{\prime \prime}(t)<0$, from $u^{[1]}(t)=\frac{1}{r(t)} u^{\prime}(t)<0$, we obtain $u^{\prime}(t)<0$, this implies $u(t)<0$. This constradiction shows that $u^{[2]}(t)>0$, thus either $u^{[1]}(t)<0$ or $u^{[1]}(t)>0$ holds, eventually. The proof is completed.
lemma 2. Assume that $x$ is a solution of (E), $u(t)$ has the proper (ii), then

$$
\begin{equation*}
\left(1-a_{0}\right)|u(t)| \leq|x(t)| \leq|u(t)| \tag{1.1}
\end{equation*}
$$

for $t \geq T$ and

$$
\begin{equation*}
\lim _{t \rightarrow \infty}|u(t)|=\lim _{t \rightarrow \infty}|x(t)|=\infty \tag{1.2}
\end{equation*}
$$

The proof of this Lemma is similar to Lemma 1 of refference of [1], and hence is ommitted.

## 2 Oscillation theorems

Theorem 1. Assume that

$$
\begin{equation*}
\int_{t_{1}}^{\infty} r(w) \int_{w}^{\infty} p(v) \int_{v}^{\infty} q(s) \mathrm{d} s \mathrm{~d} v \mathrm{~d} w=\infty \tag{2.1}
\end{equation*}
$$

Moreover, assume that $\delta(t)<t$ and there exists function $g(t)$ such that

$$
g(t) \in C\left(\left[t_{0}, \infty\right), R\right), \quad g(t)>t, \delta(g(t)) \leq t
$$

and

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \int_{t}^{g(t)} q(s) \int_{t_{0}}^{\delta(s)} r(v) \int_{t_{0}}^{v} p(w) \mathrm{d} w \mathrm{~d} v \mathrm{~d} s=\infty \tag{2.2}
\end{equation*}
$$

Then any proper solution $x$ of (E) is either oscillatory or satisfies $\lim _{t \rightarrow \infty} x(t)=0$.
Proof. Without loss of generality we may assume that $x$ is an eventually positive solution, we first assume that $u(t)$ has the proper (i). Then there exists $T_{x} \geq t_{0}$ such that $u(t)>0, u^{[1]}(t)<0, u^{[2]}(t)>0$ for $t \geq T_{x}$, we claim that

$$
\lim _{t \rightarrow \infty} u^{[i]}=l_{i}=0, i=0,1,2 .
$$

Indeed, if $l_{1}<0$, then $u^{\prime}(t) \leq l_{1} r(t)$ for large $t$,

$$
u(t) \leq u\left(T_{x}\right)+l_{1} \int_{T_{x}}^{t} r(t) \mathrm{d} t
$$

Letting $t \rightarrow \infty$, we get a contradiction with the $u(t)>0$. Therefore $l_{1}=0$. If $l_{2}>0$, then $\left(u^{[1]}(t)\right)^{\prime} \geq l_{2} p(t)$ for large $t$

$$
u^{[1]}(t) \geq u\left(t_{0}\right)+l_{2} \int_{t_{0}}^{t} p(t) \mathrm{d} t
$$

Letting $t \rightarrow \infty$, we get a contradiction with the $u^{[1]}(t)<0$. Therefore $l_{2}=0$. Assume by contradiction that $l_{0}>0$, then for any $\epsilon>0$ we have $l_{0}+\epsilon>u(\mu(t))>l_{0}$ for large $t$ and choose $0<\epsilon<\frac{l_{0}\left(1-a_{0}\right)}{a_{0}}$.

$$
\begin{equation*}
x(t)=u(t)-a(t) x(\mu(t))>l_{0}-a_{0} u(\mu(t))>l_{0}-a_{0}\left(l_{0}+\epsilon\right)=k\left(l_{0}+\epsilon\right)>k l_{0} \tag{2.3}
\end{equation*}
$$

Where $k=\frac{l_{0}-a_{0}(l+\epsilon)}{l_{0}+\epsilon}>0$. In view of the fact $f(v)$ is increasing, there exists $B>0$ such that $f(x(\delta(t))) \geq B$ for large $t$, hence frome equation (E) it follows that $\left(u^{[2]}(t)\right)^{\prime} \leq-q(t) B$. Integrating this inequality two times from $t$ to $\infty$ we obtain

$$
-u^{[1]}(t) \geq B \int_{t}^{\infty} p(v) \int_{v}^{\infty} q(s) \mathrm{d} s \mathrm{~d} v
$$

Integrating from $t_{1}$ to $t$ we obtain

$$
-u(t)+u\left(t_{1}\right) \geq B \int_{t_{1}}^{t} r(w) \int_{w}^{\infty} p(v) \int_{v}^{\infty} q(s) \mathrm{d} s \mathrm{~d} v \mathrm{~d} w
$$

Letting $t \rightarrow \infty$ we obtain

$$
\int_{t_{1}}^{\infty} r(w) \int_{w}^{\infty} p(v) \int_{v}^{\infty} q(s) \mathrm{d} s \mathrm{~d} v \mathrm{~d} w<\infty
$$

We get the contradiction with condition (2.1). Therefore $l_{0}=0$ and the inequality $0 \leq$ $x(t) \leq u(t)$ implies that $\lim _{t \rightarrow \infty} x(t)=0$.

Assume that $u(t)$ has the proper (ii). Then there exists $t_{1} \geq t_{0}$ such that $u(t)>0$, $u^{[1]}(t)>0$ and $u^{[2]}(t)>0$ for $t \geq t_{1}$, let $t_{2}$ be such that $\delta(t) \geq t_{1}$ for $t \geq t_{2}$. Because $\left(u^{[2]}(t)^{\prime}=-q(t) f(x(\delta(t)))<0\right.$ for $t \geq t_{2}, u^{[2]}(t)$ is a positive decreasing function. Integrating the equation (E) from $t$ to $\infty$ we obtain

$$
\begin{gathered}
u^{[2]}(t)=u^{[2]}(\infty)+\int_{t}^{\infty} q(s) f(x(\delta(s)) \mathrm{d} s \\
u^{[2]}(t) \geq \int_{t}^{\infty} q(s) f(x(\delta(s))) \mathrm{d} s \geq \lambda \int_{t}^{\infty} q(s) x(\delta(s)) \mathrm{d} s
\end{gathered}
$$

Using the (1.1) we obtain

$$
\begin{equation*}
\left.u^{[2]}(t) \geq \lambda\left(1-a_{0}\right) \int_{t}^{\infty} q(s) u(\delta(s)) \mathrm{d} s \geq \lambda\left(1-a_{0}\right) \int_{t}^{g(t)} q(s) u(\delta(s))\right) \mathrm{d} s \tag{2.4}
\end{equation*}
$$

Integrating $u^{[2]}(t)=u^{[2]}(t)$ twice from $t_{1}$ to $t$ we obtain

$$
u(t) \geq \int_{t_{1}}^{t} r(s) \int_{t_{1}}^{s} p(v) u^{[2]}(v) \mathrm{d} v \mathrm{~d} s
$$

for $t \geq t_{1}$, we have

$$
u(\delta(t)) \geq \int_{t_{1}}^{\delta(t)} r(s) \int_{t_{1}}^{s} p(v) u^{[2]}(v) \mathrm{d} v \mathrm{~d} s
$$

Substituting into (2.4) we get

$$
u^{[2]}(t) \geq \lambda\left(1-a_{0}\right) \int_{t}^{g(t)} q(s) \int_{t_{1}}^{\delta(s)} r(v) \int_{t_{1}}^{v} p(w) u^{[2]}(w) \mathrm{d} w \mathrm{~d} v \mathrm{~d} s
$$

Considering the fact that $u^{[2]}(t)$ is decreasing and $u^{[2]}(\delta(g(t)))$ is nonincreasing, we get

$$
u^{[2]}(t) \geq \lambda\left(1-a_{0}\right) u^{[2]}(\delta(g(t))) \int_{t}^{g(t)} q(s) \int_{t_{1}}^{\delta(s)} r(v) \int_{t_{1}}^{v} p(w) \mathrm{d} w \mathrm{~d} v \mathrm{~d} s
$$

Since $u^{[2]}(t)$ is decreasing. Lemma 1 holds, we have

$$
1 \geq \frac{u^{[2]}(t)}{u^{[2]}(\delta(g(t)))} \geq \lambda\left(1-a_{0}\right) \int_{t}^{g(t)} q(s) \int_{t_{1}}^{\delta(s)} r(v) \int_{t_{1}}^{v} p(w) \mathrm{d} w \mathrm{~d} v \mathrm{~d} s
$$

Which is contradiction of condition (2.2). The proof is completed.
Theorem 2. Assume that (2.1) and

$$
\begin{equation*}
\int_{t_{0}}^{\infty} q(t) \int_{t_{0}}^{\delta(t)} r(s) \mathrm{d} s \mathrm{~d} t=\infty \tag{2.5}
\end{equation*}
$$

Then any proper solution $x$ of (E) is either oscillatory or satisfies $\lim _{t \rightarrow \infty} x(t)=0$.
Proof. By the first part of the proof of Throrem 1 any solution $x$ tends to zero that if $u(t)=x(t)+a(t) x(\mu(t))$ has the proper (i).

Without loss of generality we may assume that $x$ is an eventually positive solution, assume that $u(t)$ has the proper (ii). Then there exists $T \geq t_{0}$ such that $u(t)>0$, $u^{[1]}(t)>0, u^{[2]}(t)>0$ for $t \geq T$. Since $u^{[1]}(t)$ is an eventually positive increasing function, we have $u^{[1]}(t)>u^{[1]}(T)$ and by integrating from $T$ to $t$ we get

$$
\begin{equation*}
u(t)>u^{[1]}(T) \int_{T}^{t} r(s) \mathrm{d} s=L \int_{T}^{t} r(s) \mathrm{d} s \tag{2.6}
\end{equation*}
$$

Using (1.1) together with (2.6) we get

$$
\begin{equation*}
x(\delta(t)) \geq u(\delta(t))\left(1-a_{0}\right) \geq\left(1-a_{0}\right) L \int_{T}^{\delta(t)} r(s) \mathrm{d} s \tag{2.7}
\end{equation*}
$$

Let $T_{1}>T$ be such that $\delta(t) \geq T_{1}$. Integrating the equation (E) from $T_{1}$ to $\infty$ we obtain

$$
u^{[2]}\left(T_{1}\right)-u^{[2]}(\infty)=\int_{T_{1}}^{\infty} q(s) f(x(\delta(s))) \mathrm{d} s
$$

Therefore $\int_{T_{1}}^{\infty} q(s) f(x(\delta(s))) \mathrm{d} s<\infty$. Since (C4) holds, we have

$$
\lambda \int_{T_{1}}^{\infty} q(s) x(\delta(s)) \mathrm{d} s \leq \int_{T_{1}}^{\infty} q(s) f(x(\delta(s))) \mathrm{d} s
$$

i.e.

$$
\lambda \int_{T_{1}}^{\infty} q(s) x(\delta(s)) \mathrm{d} s<\infty
$$

and using (2.7) we get

$$
\lambda\left(1-a_{0}\right) L \int_{T_{1}}^{\infty} q(t) \int_{T}^{\delta(t)} r(s) \mathrm{d} s \mathrm{~d} t<\infty
$$

Which contradicts (2.5). This completes the proof.
Theorem 3. Assume that $\delta(t) \leq t, f(u v) \geq f(u) f(v)$ for $u, v \in R$, (2.1) and

$$
\int_{0}^{1} \frac{1}{f(v)} \mathrm{d} v<\infty
$$

If

$$
\begin{equation*}
\int_{t_{0}}^{\infty} q(t) \int_{t_{0}}^{\delta(t)} r(s) \int_{t_{0}}^{s} p(v) \mathrm{d} v \mathrm{~d} s \mathrm{~d} t=\infty \tag{2.8}
\end{equation*}
$$

Then any proper solution $x$ of (E) is either oscillatory or satisfies $\lim _{t \rightarrow \infty} x(t)=0$.
Proof. By the first part of the proof of Throrem 1 any solution $x$ tends to zero that if $u(t)=x(t)+a(t) x(\mu(t))$ has the proper (i).

Without loss of generality we may assume that $x$ is an eventually positive solution, assume that $u(t)$ has the proper (ii). Then there exists $T \geq t_{0}$ such that $u(t)>0$, $u^{[1]}(t)>0, u^{[2]}(t)>0$ for all $t \geq T$. Because of $u^{[2]}$ is decreasing, we get

$$
u^{[1]}(t)=u^{[1]}\left(t_{1}\right)+\int_{t_{1}}^{t} p(s) u^{[2]}(s) \mathrm{d} s \geq u^{[2]}(t) \int_{t_{1}}^{t} p(s) \mathrm{d} s
$$

and therefore

$$
\begin{gather*}
u^{\prime}(t) \geq u^{[2]}(t) r(t) \int_{t_{1}}^{t} p(s) \mathrm{d} s \\
u(t) \geq u(t)-u\left(t_{1}\right)=\int_{t_{1}}^{t} u^{\prime}(s) \mathrm{d} s \geq u^{[2]}(t) \int_{t_{1}}^{t} r(s) \int_{t_{1}}^{s} p(v) \mathrm{d} v \mathrm{~d} s \tag{2.9}
\end{gather*}
$$

Using (E) and (1.1) we get

$$
-\left(u^{[2]}(t)\right)^{\prime}=q(t) f(x(\delta(t))) \geq q(t) f\left(1-a_{0}\right) f(u(\delta(t)))
$$

Using (C4) and ((2.9) we get

$$
-\left(u^{[2]}(t)\right)^{\prime} \geq \lambda q(t) f\left(1-a_{0}\right) f\left(u^{[2]}(t)\right) \int_{t_{1}}^{\delta(t)} r(s) \int_{t_{1}}^{s} p(v) \mathrm{d} v \mathrm{~d} s
$$

Hence

$$
-\int_{t_{1}}^{t} \frac{\left.u^{[2]}(t)\right)^{\prime}}{f\left(u^{[2]}(t)\right)} \mathrm{d} t \geq \lambda f\left(1-a_{0}\right) \int_{t_{1}}^{t} q(w) \int_{t_{1}}^{\delta(w)} r(s) \int_{t_{1}}^{s} p(v) \mathrm{d} v \mathrm{~d} s \mathrm{~d} w
$$

Letting $t \rightarrow \infty$

$$
-\int_{t_{1}}^{\infty} \frac{\left.u^{[2]}(t)\right)^{\prime}}{f\left(u^{[2]}(t)\right)} \mathrm{d} t=\int_{u^{[2]}(\infty)}^{u^{[2]}\left(t_{1}\right)} \frac{d s}{f(s)}<\infty
$$

We get the contradiction with condition (2.8). The proof is completed.

## 3 Examples

Example 1. Consider the equation

$$
\begin{equation*}
\left(\frac{1}{t}\left[x(t)+\frac{1}{3 t} x\left(\frac{t}{2}\right)\right]^{\prime}\right)^{\prime \prime}+\frac{1}{t^{3}} x\left(k^{2} t\right)=0, \quad t \geq 1 \tag{3.1}
\end{equation*}
$$

where $0<k<1$. If we take $g(t)=\frac{t}{k}$. One can check that condition (2.1) and (2.2) are satisfied. Thus, by Theorem 1, then any proper solution $x$ of (3.1) is either oscillatory or satisfies $\lim _{t \rightarrow \infty} x(t)=0$.

Example 2. Consider the equation

$$
\begin{equation*}
\left(\frac{1}{t}\left[x(t)+\frac{1}{5 t} x\left(\frac{t}{2}\right)\right]^{\prime}\right)^{\prime \prime}+\frac{1}{t^{3}} x\left(\frac{t}{3}\right)=0, \quad t \geq 1 \tag{3.2}
\end{equation*}
$$

One can check that condition (2.1) and (2.5) are satisfied. Thus, by Theorem 2, then any proper solution $x$ of (3.2) is either oscillatory or satisfies $\lim _{t \rightarrow \infty} x(t)=0$.

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