Oscillation criteria for a class of third-order differential equations with neutral term

Abstract

This paper is concerned with oscillation criteria for a class of third-order differential equations with neutral term by using some necessary analysis techniques, some sufficient conditions for oscillation are obtained, some examples are provided to illustrate the main results.

Keywords: Oscillation; Neutral; Third-order; Differential equations 2010 AMS Subject Classification: 41A25, 65D99

1 Introduction

In this paper, we consider the oscillatory and asymptotic properties for a class of thirdorder nonlinear differential equation with damped term

$$\left(\frac{1}{p(t)} \left(\frac{1}{r(t)} \left[x(t) + a(t)x(\mu(t))\right]'\right)'\right)' + q(t)f(x(\delta(t))) = 0, \quad t \ge t_0$$
(E)

As usual, we use the notation, $u(t) = x(t) + a(t)x(\mu(t))$. In what follows, it is always assume

- (C1) $p(t), r(t), a(t), q(t), \delta(t), \mu(t) \in C([t_0, \infty), (0, \infty)),$
- (C2) $\int_{t_0}^{\infty} p(t) dt = \int_{t_0}^{\infty} r(t) dt = \infty, r'(t) > 0,$
- (C3) $\mu(t) \le t$, $\lim_{t\to\infty} \mu(t) = \lim_{t\to\infty} \delta(t) = \infty$,
- (C4) $0 \le a(t) \le a_0 < 1$, $f \in C(R, R)$, f'(v) > 0, $\frac{f(v)}{v} \ge \lambda$, for all $v \ne 0$, and for some $\lambda > 0$.

By a solution of equation (E) we mean a continuous function x(t) definned on an interval $[t_0, \infty)$ such that $\left(\frac{1}{r(t)} \left[x(t) + a(t)x(\mu(t))\right]'\right)'$ is continuously differentiable satifies (E), we assum that equation (E) have such solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[t_0, \infty)$, otherwise, it is called nonoscillatory. We say equation (E) is oscillatory if all its continuable solutions are oscillatory.

In what follows, we consider only proper solution of the equation (E) which are defined for all large t. More and more people are interested in oscillatory and nonoscillatory criteria to be shown^[1-7]. Our principal goal in this paper is to derive new oscillation criteria for equation (E), without requiring restrictive condition (4) and (7) in [1]

For simplicity, we introduce the following nonation:

$$u^{[0]}(t) = u(t), \quad u^{[1]}(t) = \frac{1}{r(t)}u'(t), \quad u^{[2]}(t) = \frac{1}{p(t)}(u^{[1]}(t))'$$

lemma 1. Let x(t) be a nonscillatory solution of (E), then there exists a T_x for $t > T_x \ge t_0$, such that u(t) has only the following two cases.

- (i) $u(t)u^{[1]}(t) < 0, \quad u(t)u^{[2]}(t) > 0,$
- (ii) $u(t)u^{[1]}(t) > 0$, $u(t)u^{[2]}(t) > 0$.

Proof. Without loss of generality we may assume that x(t) is eventually positive, i.e. there exists $T_x \ge t_0$ such that x(t) > 0, u(t) > 0 for $t \ge T_x$. (If it is an eventually negative ,the proof is similar). Using (E) we get $(u^{[2]}(t))' < 0$, eventually. Then $u^{[2]}(t)$ is decreasing and of one sign for $t \ge T_x$. If we admit $u^{[2]}(t) < 0$, then there exists a constant M > 0such that

$$\frac{1}{p(t)}u^{[1]}(t))' \le -M < 0,$$
$$(u^{[1]}(t))' \le -Mp(t)$$

Integrating from T_x to t, we obtain

$$u^{[1]}(t) \le u^{[1]}(T_x) - M \int_{T_x}^t p(s) \mathrm{d}s$$

Leting $t \to \infty$ and using (C2), we get $u^{[1]}(t) < 0$, which together with r'(t) > 0 and $u^{[2]}(t) = \frac{r(t)u''(t) - r'(t)u'(t)}{p(t)r^2(t)} < 0$. We get u''(t) < 0, from $u^{[1]}(t) = \frac{1}{r(t)}u'(t) < 0$, we obtain u'(t) < 0, this implies

We get u''(t) < 0, from $u^{[1]}(t) = \frac{1}{r(t)}u'(t) < 0$, we obtain u'(t) < 0, this implies u(t) < 0. This constradiction shows that $u^{[2]}(t) > 0$, thus either $u^{[1]}(t) < 0$ or $u^{[1]}(t) > 0$ holds, eventually. The proof is completed.

lemma 2. Assume that x is a solution of (E), u(t) has the proper (ii), then

$$(1 - a_0)|u(t)| \le |x(t)| \le |u(t)|, \tag{1.1}$$

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for $t \geq T$ and

$$\lim_{t \to \infty} |u(t)| = \lim_{t \to \infty} |x(t)| = \infty$$
(1.2)

The proof of this Lemma is similar to Lemma 1 of refference of [1], and hence is ommitted.

2 Oscillation theorems

Theorem 1. Assume that

$$\int_{t_1}^{\infty} r(w) \int_{w}^{\infty} p(v) \int_{v}^{\infty} q(s) \mathrm{d}s \mathrm{d}v \mathrm{d}w = \infty$$
(2.1)

Moreover, assume that $\delta(t) < t$ and there exists function g(t) such that

$$g(t) \in C([t_0, \infty), R), \quad g(t) > t, \ \delta(g(t)) \le t,$$

and

$$\lim_{t \to \infty} \int_{t}^{g(t)} q(s) \int_{t_0}^{\delta(s)} r(v) \int_{t_0}^{v} p(w) \mathrm{d}w \mathrm{d}v \mathrm{d}s = \infty$$
(2.2)

Then any proper solution x of (E) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Without loss of generality we may assume that x is an eventually positive solution, we first assume that u(t) has the proper (i). Then there exists $T_x \ge t_0$ such that u(t) > 0, $u^{[1]}(t) < 0$, $u^{[2]}(t) > 0$ for $t \ge T_x$, we claim that

$$lim_{t\to\infty}u^{[i]} = l_i = 0, i = 0, 1, 2.$$

Indeed, if $l_1 < 0$, then $u'(t) \le l_1 r(t)$ for large t,

$$u(t) \le u(T_x) + l_1 \int_{T_x}^t r(t) \mathrm{d}t$$

Letting $t \to \infty$, we get a contradiction with the u(t) > 0. Therefore $l_1 = 0$. If $l_2 > 0$, then $(u^{[1]}(t))' \ge l_2 p(t)$ for large t

$$u^{[1]}(t) \ge u(t_0) + l_2 \int_{t_0}^t p(t) \mathrm{d}t$$

Letting $t \to \infty$, we get a contradiction with the $u^{[1]}(t) < 0$. Therefore $l_2 = 0$. Assume by contradiction that $l_0 > 0$, then for any $\epsilon > 0$ we have $l_0 + \epsilon > u(\mu(t)) > l_0$ for large t and choose $0 < \epsilon < \frac{l_0(1-a_0)}{a_0}$.

$$x(t) = u(t) - a(t)x(\mu(t)) > l_0 - a_0 u(\mu(t)) > l_0 - a_0(l_0 + \epsilon) = k(l_0 + \epsilon) > kl_0$$
(2.3)

Where $k = \frac{l_0 - a_0(l + \epsilon)}{l_0 + \epsilon} > 0$. In view of the fact f(v) is increasing, there exists B > 0 such that $f(x(\delta(t))) \ge B$ for large t, hence from equation (E) it follows that $(u^{[2]}(t))' \le -q(t)B$. Integrating this inequality two times from t to ∞ we obtain

$$-u^{[1]}(t) \ge B \int_t^\infty p(v) \int_v^\infty q(s) \mathrm{d}s \mathrm{d}v$$

Integrating from t_1 to t we obtain

$$-u(t) + u(t_1) \ge B \int_{t_1}^t r(w) \int_w^\infty p(v) \int_v^\infty q(s) \mathrm{d}s \mathrm{d}v \mathrm{d}w$$

Letting $t \to \infty$ we obtain

$$\int_{t_1}^{\infty} r(w) \int_{w}^{\infty} p(v) \int_{v}^{\infty} q(s) \mathrm{d}s \mathrm{d}v \mathrm{d}w < \infty$$

We get the contradiction with condition (2.1). Therefore $l_0 = 0$ and the inequality $0 \le x(t) \le u(t)$ implies that $\lim_{t\to\infty} x(t) = 0$.

Assume that u(t) has the proper (ii). Then there exists $t_1 \ge t_0$ such that u(t) > 0, $u^{[1]}(t) > 0$ and $u^{[2]}(t) > 0$ for $t \ge t_1$, let t_2 be such that $\delta(t) \ge t_1$ for $t \ge t_2$. Because $(u^{[2]}(t)' = -q(t)f(x(\delta(t))) < 0$ for $t \ge t_2$, $u^{[2]}(t)$ is a positive decreasing function. Integrating the equation (E) from t to ∞ we obtain

$$\begin{split} u^{[2]}(t) &= u^{[2]}(\infty) + \int_t^\infty q(s) f(x(\delta(s)) \mathrm{d}s \\ u^{[2]}(t) &\geq \int_t^\infty q(s) f(x(\delta(s))) \mathrm{d}s \geq \lambda \int_t^\infty q(s) x(\delta(s)) \mathrm{d}s \end{split}$$

Using the (1.1) we obtain

$$u^{[2]}(t) \ge \lambda(1 - a_0) \int_t^\infty q(s) u(\delta(s)) ds \ge \lambda(1 - a_0) \int_t^{g(t)} q(s) u(\delta(s))) ds$$
(2.4)

Integrating $u^{[2]}(t) = u^{[2]}(t)$ twice from t_1 to t we obtain

$$u(t) \ge \int_{t_1}^t r(s) \int_{t_1}^s p(v) u^{[2]}(v) \mathrm{d}v \mathrm{d}s$$

for $t \geq t_1$, we have

$$u(\delta(t)) \ge \int_{t_1}^{\delta(t)} r(s) \int_{t_1}^s p(v) u^{[2]}(v) \mathrm{d}v \mathrm{d}s$$

Substituting into (2.4) we get

$$u^{[2]}(t) \ge \lambda(1-a_0) \int_t^{g(t)} q(s) \int_{t_1}^{\delta(s)} r(v) \int_{t_1}^v p(w) u^{[2]}(w) \mathrm{d}w \mathrm{d}v \mathrm{d}s$$

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Considering the fact that $u^{[2]}(t)$ is decreasing and $u^{[2]}(\delta(g(t)))$ is nonincreasing, we get

$$u^{[2]}(t) \ge \lambda(1 - a_0)u^{[2]}(\delta(g(t))) \int_t^{g(t)} q(s) \int_{t_1}^{\delta(s)} r(v) \int_{t_1}^v p(w) \mathrm{d}w \mathrm{d}v \mathrm{d}s$$

Since $u^{[2]}(t)$ is decreasing. Lemma 1 holds, we have

$$1 \ge \frac{u^{[2]}(t)}{u^{[2]}(\delta(g(t)))} \ge \lambda(1-a_0) \int_t^{g(t)} q(s) \int_{t_1}^{\delta(s)} r(v) \int_{t_1}^v p(w) \mathrm{d}w \mathrm{d}v \mathrm{d}s$$

Which is contradiction of condition (2.2). The proof is completed.

Theorem 2. Assume that (2.1) and

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{\delta(t)} r(s) \mathrm{d}s \mathrm{d}t = \infty$$
(2.5)

Then any proper solution x of (E) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. By the first part of the proof of Throrem 1 any solution x tends to zero that if $u(t) = x(t) + a(t)x(\mu(t))$ has the proper (i).

Without loss of generality we may assume that x is an eventually positive solution, assume that u(t) has the proper (ii). Then there exists $T \ge t_0$ such that u(t) > 0, $u^{[1]}(t) > 0$, $u^{[2]}(t) > 0$ for $t \ge T$. Since $u^{[1]}(t)$ is an eventually positive increasing function, we have $u^{[1]}(t) > u^{[1]}(T)$ and by integrating from T to t we get

$$u(t) > u^{[1]}(T) \int_{T}^{t} r(s) ds = L \int_{T}^{t} r(s) ds$$
 (2.6)

Using (1.1) together with (2.6) we get

$$x(\delta(t)) \ge u(\delta(t))(1-a_0) \ge (1-a_0)L \int_T^{\delta(t)} r(s) \mathrm{d}s$$
 (2.7)

Let $T_1 > T$ be such that $\delta(t) \ge T_1$. Integrating the equation (E) from T_1 to ∞ we obtain

$$u^{[2]}(T_1) - u^{[2]}(\infty) = \int_{T_1}^{\infty} q(s) f(x(\delta(s))) ds$$

Therefore $\int_{T_1}^{\infty} q(s) f(x(\delta(s))) ds < \infty$. Since (C4) holds, we have

$$\lambda \int_{T_1}^{\infty} q(s) x(\delta(s)) \mathrm{d}s \le \int_{T_1}^{\infty} q(s) f(x(\delta(s))) \mathrm{d}s$$

i.e.

$$\lambda \int_{T_1}^{\infty} q(s) x(\delta(s)) \mathrm{d} s < \infty$$

and using (2.7) we get

$$\lambda(1-a_0)L\int_{T_1}^{\infty}q(t)\int_{T}^{\delta(t)}r(s)\mathrm{d}s\mathrm{d}t<\infty$$

Which contradicts
$$(2.5)$$
. This completes the proof.

Theorem 3. Assume that $\delta(t) \leq t$, $f(uv) \geq f(u)f(v)$ for $u, v \in R$, (2.1) and

$$\int_0^1 \frac{1}{f(v)} \mathrm{d}v < \infty$$

If

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{\delta(t)} r(s) \int_{t_0}^{s} p(v) \mathrm{d}v \mathrm{d}s \mathrm{d}t = \infty$$
(2.8)

Then any proper solution x of (E) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. By the first part of the proof of Throrem 1 any solution x tends to zero that if $u(t) = x(t) + a(t)x(\mu(t))$ has the proper (i).

Without loss of generality we may assume that x is an eventually positive solution, assume that u(t) has the proper (ii). Then there exists $T \ge t_0$ such that u(t) > 0, $u^{[1]}(t) > 0$, $u^{[2]}(t) > 0$ for all $t \ge T$. Because of $u^{[2]}$ is decreasing, we get

$$u^{[1]}(t) = u^{[1]}(t_1) + \int_{t_1}^t p(s)u^{[2]}(s) \mathrm{d}s \ge u^{[2]}(t) \int_{t_1}^t p(s) \mathrm{d}s$$

and therefore

$$u'(t) \ge u^{[2]}(t)r(t)\int_{t_1}^t p(s)ds$$
$$u(t) \ge u(t) - u(t_1) = \int_{t_1}^t u'(s)ds \ge u^{[2]}(t)\int_{t_1}^t r(s)\int_{t_1}^s p(v)dvds$$
(2.9)

Using (E) and (1.1) we get

$$-(u^{[2]}(t))' = q(t)f(x(\delta(t))) \ge q(t)f(1-a_0)f(u(\delta(t)))$$

Using (C4) and ((2.9) we get

$$-(u^{[2]}(t))' \ge \lambda q(t)f(1-a_0)f(u^{[2]}(t))\int_{t_1}^{\delta(t)} r(s)\int_{t_1}^s p(v)\mathrm{d}v\mathrm{d}s$$

Hence

$$-\int_{t_1}^t \frac{u^{[2]}(t))'}{f(u^{[2]}(t))} \mathrm{d}t \ge \lambda f(1-a_0) \int_{t_1}^t q(w) \int_{t_1}^{\delta(w)} r(s) \int_{t_1}^s p(v) \mathrm{d}v \mathrm{d}s \mathrm{d}w$$

Letting $t \to \infty$

$$-\int_{t_1}^{\infty} \frac{u^{[2]}(t))'}{f(u^{[2]}(t))} \mathrm{d}t = \int_{u^{[2]}(\infty)}^{u^{[2]}(t_1)} \frac{\mathrm{d}s}{f(s)} < \infty$$

We get the contradiction with condition (2.8). The proof is completed.

3 Examples

Example 1. Consider the equation

$$\left(\frac{1}{t}[x(t) + \frac{1}{3t}x(\frac{t}{2})]'\right)'' + \frac{1}{t^3}x(k^2t) = 0, \quad t \ge 1$$
(3.1)

where 0 < k < 1. If we take $g(t) = \frac{t}{k}$. One can check that condition (2.1) and (2.2) are satisfied. Thus, by Theorem 1, then any proper solution x of (3.1) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Example 2. Consider the equation

$$\left(\frac{1}{t}[x(t) + \frac{1}{5t}x(\frac{t}{2})]'\right)'' + \frac{1}{t^3}x(\frac{t}{3}) = 0, \quad t \ge 1$$
(3.2)

One can check that condition (2.1) and (2.5) are satisfied. Thus, by Theorem 2, then any proper solution x of (3.2) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

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