Original Research Article

Accurate time calculations of falling bodies in the Earth's gravitational field and comparisons with Newton's laws of vertical motion

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Abstract

The Earth is exposed annually to the fall of some meteorites and probably other celestial bodies. This event causes a potential danger to vital areas in several countries. Consequently, the accurate calculation of the falling times of such bodies is, in general, useful to take the necessary procedures to protect these areas in view of the calculated falling time. Several centuries ago, the British scientist Isaac Newton developed the laws of regular motion with a constant acceleration in a straight line. Such laws are often studied in the early years of the university stage to investigate the vertical motion of objects close enough to the surface of the earth, that is, at small heights compared to the radius of the Earth. Newton also discovered his important law of general gravitation in classical mechanics, which is usually used to analyze the motion of an object in the gravitational field of another object. The latter is of course more general than the aforementioned vertical motion laws. The question that we want to answer in the present study is that; what is the difference between the falling time of an object in view of both Newton's laws of vertical motion and Newton's law of general gravitation? In the present study, we will determine the amount of error resulting from the applications of Newton's laws of vertical motion. Such an error will be expressed in terms of the height from which an object fall. The results are applied on several bodies in real life and the obtained errors are tabulated.

Keywords: Newton's laws; vertical motion; general gravitation; falling time.

1 Introduction

Perhaps the first attempt to study the falling objects in the Earth's gravitational field was the experiment of the pound and the quill made by the great scientist Isaac Newton several centuries ago. Newton concluded from his experiment that the falling time of two bodies from the same height does not depend on their masses and that they will take the same time to reach the ground in the absence of air resistance, where Newton has conducted his famous experiment in a vacuum tube of air. Newton made great contributions and discovered many scientific laws in classical mechanics [1], not only, he also developed other important theories and laws in various branches of physics and astronomy. Some of the most famous laws developed by Newton were the three laws of motion in a straight line with a constant/regular acceleration. These laws are usually taught in the early years for the students in physics and mathematics departments. Replacing the constant acceleration by the acceleration due to the gravity of Earth leads to Newton's laws of vertical motion which are used to study the vertical motion of objects near to the Earth's surface.

The question arises here is that; is it possible to apply Newton's laws of vertical motion on objects falling from hundreds of kilometers above the ground? On the other hand, Newton derived his famous law of gravitation through which can be used to study the vertical motion of objects far away from the Earth's surface [2]. Hence, Newton's law of gravitation could be applied in a much greater range than Newton's laws of vertical motion. The questions that we try to answer in the current study are; what is the difference between the results that can be obtained in light of both Newton's laws of vertical motion and Newton's law of gravitation? Is the falling time derived from both types of laws will be the same when an object falls from a prescribed height above the ground? In the present research, we will be able to determine the amount of error resulting from the applications of Newton's laws of vertical motion in estimating the falling time of objects as a function of the height. The equation of motion of a particle in a general resistant medium in view of Newton's law of gravitation is given by [2]

$$\ddot{r}(t) = -\frac{GM}{r^2} + kv^n, \tag{1}$$

where, M is the mass of Earth, G is Newton's constant of general gravitation, r is the distance of the particle from the center of Earth, $v(t) = \dot{r}(t)$ is the vertical instantaneous velocity of the particle, k is the constant of resistance, and n is a positive natural number. Assuming that R is the radius of Earth, the initial conditions (ICs) are given as

$$\dot{r}(0) = 0, \qquad r(0) = h + R \tag{2}$$

where h is the height of the particle above the Earth's surface at initial time. The proposed method depends basically on applying some basic concepts in calculus [3,4] for the special case $k = 0$. In case $k \neq 0, n \geq 1$, the Adomian decomposition method (ADM) [5-14] may be applied to solve the nonlinear system (1-2) which is a complex nonlinear initial value problem. The objectives of this paper are focused on estimating the falling time and comparing our results with the corresponding ones obtained from Newton's laws of vertical motion. Then, applying the results on several bodies in our real life.

2 The exact solution

In this section, the exact solution of the system (1-2) will be obtained at the special case $k = 0$. In this case, Eq. (1) becomes

$$\ddot{r}(t) = -\frac{GM}{r^2}. \tag{3}$$

Multiplying both sides by $\dot{r}$, we have

$$\dot{r}\ddot{r}(t) = -GMr^{-2}\dot{r}. \tag{4}$$

Integrating once, yields

$$\frac{1}{2}\left(\dot{r}(t)\right)^2 = \frac{GM}{r} + c_1, \tag{5}$$

where c_1 is a constant of integration. Applying the ICs (2), we obtain

$$c_1 = -\frac{GM}{h+R}, \tag{6}$$

and Eq. (5) becomes

$$\left(\dot{r}(t)\right)^2 = 2GM\left(\frac{1}{r} - \frac{1}{h+R}\right), \tag{7}$$

or

$$\dot{r}(t) = \pm\sqrt{2GM\left(\frac{1}{r} - \frac{1}{h+R}\right)} \tag{8}$$

Since $r(t)$ is a decreasing function in time, we choose the negative sign in (8) and this yields

$$\frac{dr}{dt} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{h+R}\right)}. \tag{9}$$

Making use of the new variable $u = \frac{1}{r}$, we have

$$\frac{dr}{dt} = -\frac{1}{u^2}\,\frac{du}{dt}. \tag{10}$$

Inserting (10) into (9) gives

$$-\frac{1}{u^2}\,\frac{du}{dt} = -\sqrt{2GM\,(u-\alpha)}, \tag{11}$$

where

$$\alpha = \frac{1}{h+R}. \tag{12}$$

Using separation of variables approach, we can write (11) as

$$\frac{du}{u^2\sqrt{u-\alpha}} = \sqrt{2GM}\;dt. \tag{13}$$

Integrating once again, yields

$$\int \frac{du}{u^2\sqrt{u-\alpha}} = \sqrt{2GM}\;t + c_2, \tag{14}$$

where c_2 is also a constant of integration. Implementing the trigonometric substitution method, we have

$$u = \alpha\,(\sec\phi)^2, \qquad du = 2\alpha\,(\sec\phi)^2\tan\phi d\phi. \tag{15}$$

Substituting (15) into (14) and simplifying leads to

$$\frac{2}{\sqrt{\alpha^3}}\int (\cos\phi)^2 d\phi = \sqrt{2GM}\;t + c_2, \tag{16}$$

i.e.,

$$\frac{1}{\sqrt{\alpha^3}}\,(\phi + \sin\phi\cos\phi) = \sqrt{2GM}\;t + c_2. \tag{17}$$

From (15), we have

$$\phi = \sec^{-1}\left(\sqrt{\frac{u}{\alpha}}\right), \qquad \cos\phi = \sqrt{\frac{\alpha}{u}}, \qquad \sin\phi = \sqrt{1-\frac{\alpha}{u}}. \tag{18}$$

Hence,

$$\frac{1}{\sqrt{\alpha^3}}\left[\sec^{-1}\left(\sqrt{\frac{u}{\alpha}}\right) + \sqrt{\frac{\alpha}{u}}\sqrt{1-\frac{\alpha}{u}}\right] = \sqrt{2GM}\;t + c_2, \tag{19}$$

or

$$\sqrt{(h+R)^3}\left[\sec^{-1}\left(\sqrt{\frac{u}{\alpha}}\right) + \sqrt{\frac{\alpha}{u}}\sqrt{1-\frac{\alpha}{u}}\right] = \sqrt{2GM}\;t + c_2, \tag{20}$$

From the ICs (2), we have

$$u(0) = \frac{1}{r(0)} = \frac{1}{h+R} = \alpha, \tag{21}$$

Applying this condition on Eq. (20), we obtain

$$c_2 = \sqrt{(h+R)^3}\,\sec^{-1}(1) = 0, \tag{22}$$

and hence,

$$\sqrt{(h+R)^3}\left[\sec^{-1}\left(\sqrt{\frac{u}{\alpha}}\right)+\sqrt{\frac{\alpha}{u}}\sqrt{1-\frac{\alpha}{u}}\right]=\sqrt{2GM}\ t. \tag{23}$$

The product GM is also given by

$$GM = gR^2, \tag{24}$$

where R is the radius of the Earth and g is the acceleration due to gravity of the Earth. Accordingly, the equation (22) gives the falling time as

$$t=\sqrt{\frac{(h+R)^3}{2gR^2}}\left[\sec^{-1}\left(\sqrt{\frac{u}{\alpha}}\right)+\sqrt{\frac{\alpha}{u}}\sqrt{1-\frac{\alpha}{u}}\right], \tag{25}$$

or in terms of r and h as

$$t=\sqrt{\frac{(h+R)^3}{2gR^2}}\left[\sec^{-1}\left(\sqrt{\frac{h+R}{r}}\right)+\sqrt{\frac{r}{h+R}}\sqrt{1-\frac{r}{h+R}}\right]. \tag{26}$$

The object reaches the Earth's surface when $r = R$ and accordingly the exact falling time T_{Exact} is expressed as

$$T_{\text{Exact}}=\sqrt{\frac{(h+R)^3}{2gR^2}}\left[\sec^{-1}\left(\sqrt{\frac{h+R}{R}}\right)+\sqrt{\frac{R}{h+R}}\sqrt{1-\frac{R}{h+R}}\right]. \tag{27}$$

Newton's laws of vertical motion for a falling object are well-Known as

$$v = v_0 + gt, \qquad v_0 = \dot{r}(0) = 0, \tag{28}$$

$$h = v_0 t + \frac{1}{2}gt^2, \tag{29}$$

$$v^2 = v_0^2 + 2gh. \tag{30}$$

From Eq. (30), the approximate falling time T_{Approx} is expressed as

$$T_{\text{Approx}} = \sqrt{\frac{2h}{g}}. \tag{31}$$

Therefore, the absolute error in estimating the falling time, in terms of the height h, $\text{Error}(h)$, is given by

$$\text{Error}(h) = T_{\text{Exact}} - T_{\text{Approx}}, \tag{32}$$

i.e.,

$$\text{Error}(h)=\sqrt{\frac{(h+R)^3}{2gR^2}}\left[\sec^{-1}\left(\sqrt{\frac{h+R}{R}}\right)+\sqrt{\frac{R}{h+R}}\sqrt{1-\frac{R}{h+R}}\right]-\sqrt{\frac{2h}{g}}. \tag{33}$$

3 Applications

In this section, we give some applications of the exact formula of the falling time for various objects in real life. The following approximate values of the radius of Earth and the acceleration due to gravity of Earth are implemented to conduct the results of this section:

$$R \approx 6400 \times 10^3 \text{ [meter]}, \qquad g \approx 10 \text{ [meter}/s^2\text{]}. \tag{34}$$

3.1 Aeroplane

It is well known in the fields of aviation and air transport that the aircrafts, which are used in internal or international flights, fly at altitudes ranging between 29000 and 35000 feet, equivalent to 9 to 11 kilometers above the Earth's surface. Perhaps the reason for flying at this altitude is that such a layer of the Earth's atmosphere is more stable than other layers. We are now facing the question that if all of the plane's engines are suddenly stopped, how long will it take for them to fall on the ground? To answer that question, let us consider that the average altitude of aircraft is 10 kilometers above the Earth's surface, and by substituting for $h = 10$ [Km] or $h = 10000$ [m] in Eq. (27) we obtain $T_{\text{Exact}} = 44.7796 \approx 45$ seconds, which means that the aircraft takes about 45 seconds to reach the ground in the absence of air resistance. This may coincide with the same time when the atomic bomb fell on the cities of Hiroshima and Nagasaki in Japan during the second world war. Applying the approximate formula (31), we have $T_{\text{Approx}} = 44.7214 \approx 45$ seconds. The error (32) in this case is too small and given as $\text{Error} = T_{\text{Exact}} - T_{\text{Approx}} = 44.7796 - 44.7214 = 0.0582$ seconds This is because the height $h = 10$ [Km] is a very small height if compared with the radius of the Earth.

3.2 Geostationary Satellites

The geostationary satellites are at altitude $h = 36000$ [Km] above the Earth's surface and they are in stationary orbits around the Earth. Assume that the motion of such satellites is suddenly stopped, regardless of how this happens, then the expected exact time taken by these satellites to reach the Earth's surface is calculated from (27) as $T_{\text{Exact}} = 14756$ seconds. Converting this value into hours and minutes gives $T_{\text{Exact}} = 4$ hours and 6 minutes.

3.3 The Moon around Earth

The Moon is at a distance 384400 Km from the center of the Earth, consequently, the corresponding height is $h = 384400 - 6400 = 378000$ [Km]. Substituting $h = 378000$

[Km] in (27), we get $T_{\text{Exact}} = 413239$ seconds. Converting this value into days and hours gives $T_{\text{Exact}} = 4$ days and 18 hours. This result given the falling time of the Moon on the Earth (assuming that the Moon is suddenly stopped, whatever the reason) agrees with the obtained result in Ref. [15] (Problem 5.107, page 141).

3.4 The Earth around Sun

The Earth is at a distance 150×10^6 Km from the center of the Sun. In order to estimate the falling time of the Earth on the Sun (assuming that the Earth is suddenly stopped, whatever the reason), we modify Eq. (27) as

$$T_{\text{Exact}} = \sqrt{\frac{(h+R_s)^3}{2g_s R_s^2}} \left[\sec^{-1}\left(\sqrt{\frac{h+R_s}{R_s}}\right) + \sqrt{\frac{R_s}{h+R_s}}\sqrt{1-\frac{R_s}{h+R_s}} \right], \tag{35}$$

where R_s is the radius of the Sun ($R_s = 6.96 \times 10^5$ Km) and g_s is the acceleration of gravity due to the Sun ($g_s = 273$ [meter/s^2]). In this case, the height of Earth above the Sun equals $h = 150 \times 10^6 - 6.96 \times 10^5 = 149.304 \times 10^6$ [Km]. Accordingly, Eq. (35) leads to $T_{\text{Exact}} = 5.61039 \times 10^6$ seconds. Converting this value into days gives $T_{\text{Exact}} = 64.9 \approx 65$ days. Also, this result agrees with the obtained result in Ref. [15] (Problem 5.108, page 141).

4 Discussion of errors

Using the error equation (33), we present in Tables (1-3) some numerical results, from which it becomes clear that the amount of error in time is about 20 seconds in the first 500 kilometers above the surface of the earth, as in Table (1).

Table 1: Calculated errors for $h = 100$, 200, 300, 400, 500 Km.

h [Km]	100	200	300	400	500
Error(h)[S]	1.84057	5.20355	9.55526	14.7048	20.5418

While the error in time is about 58 seconds or approximately one minute, in the first 1000 km, as shown in Table (2).

Table 2: Calculated errors for $h =$ 600, 700, 800, 900, 1000 Km.

h[Km]	600	700	800	900	1000
Error(h)[S]	26.9915	33.999	41.5219	49.5259	57.9827

It is clear from Table (3) that the amount of error in time is approximately 840 seconds, that is, 14 minutes if the altitude reaches six thousand kilometers.

Table 3: Calculated errors for $h=$ 2000, 3000, 4000, 5000, 6000Km.

h[Km]	2000	3000	4000	5000	6000
Error(h)[S]	163.412	299.287	459.555	640.747	840.537

In addition, it can be indicated from Eq. (33) that the error in time is approximately twelve thousand seconds, i.e. about three hours and a third of an hour if the body fell from a height of 36,000 km above the surface of the Earth, which is the same height as the motion of geosynchronous satellites. In light of these results, it becomes clear to us that it is preferable not to apply Newton's laws of vertical motion at altitudes higher than a thousand kilometers above the surface of the Earth, as the amount of error in the estimated time of falling objects becomes minutes and increases to hours with the increase in height of the object above the surface of the Earth.

5 Conclusion

In this paper, Newton's law of general gravitation was applied to analyze the vertical motion of an object towards the Earth. The exact falling time formula is obtained explicitly and such a formula was invested to calculate the falling time of some objects in our real life. The results revealed that the time taken by a plane, if all of the plane's engines are suddenly stopped, to reach the ground was about 45 seconds in the absence of air resistance. The geostationary satellites, which were at altitude $h = 36000$ [Km] above the Earth's surface, reach the Earth's surface in 4 hours and 6 minutes, under the assumption that the motion of such satellites is suddenly stopped, regardless of how this happens. In addition and under such assumption, the Moon takes about 4 days and 18 hours to fall on the Earth. Furthermore, the time taken by the Earth to reach the Sun's surface was about 65 days. The last two results were in full agreement with the calculations in Ref. [15] (Problems 5.107 and 5.108, page 141). Finally, the amount of error resulting from the applications of Newton's laws of vertical motion was obtained and expressed in terms of the height. In view of the obtained results, it was recommended to avoid the use of Newton's laws of vertical motion at altitudes higher than a thousand kilometers above the surface of the Earth. This is because as the amount of error in the estimated time of falling objects becomes minutes and increases to hours by increasing the height of the object above the surface of the Earth.

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